Алгебра и анализ
Том 30 (2018), №6

NETS OF GRADED C^*-ALGEBRAS OVER PARTIALLY ORDERED SETS

© S. A. GRIGORYAN, E. V. LIPACHEVA, A. S. SITDIKOV

The paper deals with C^*-algebras generated by a net of Hilbert spaces over a partially ordered set. The family of those algebras constitutes a net of C^*-algebras over the same set. It is shown that every such an algebra is graded by the first homotopy group of the partially ordered set. We consider inductive systems of C^*-algebras and their limits over maximal directed subsets. We also study properties of morphisms for nets of Hilbert spaces as well as nets of C^*-algebras.

§1. Introduction

The paper is devoted to the construction and the study of nets consisting of C^*-algebras generated by nets of Hilbert spaces over partially ordered sets. One of the directions in the applications of such nets is algebraic quantum field theory. In a spacetime the family of all open bounded regions is a partially ordered set under the inclusion relation [1, 2, 3]. One associates to these regions the C^*-algebras of local observables which can be measured in the pertinent regions. The family of all those algebras indexed by the regions of a spacetime is called a net of C^*-algebras.

The main objects of the study in the paper are the local C^*-algebras (see § 5) generated by a triple

$$\big(K, H_a, \gamma_{ba}\big)_{a\leqslant b\in K},$$

where K is a partially ordered set, H_a is a Hilbert space and $\gamma_{ba} : H_a \to H_b$ is an isometric embedding. According to papers [4, 5, 6], we call that triple a net of Hilbert spaces over K. The family of the C^*-algebras constitutes a net over the same set K. Each algebra in this net is graded by the first homotopy group $\pi_1(K)$ for the partially ordered set K.

Moreover, we introduce the notion of the corona for a net consisting of the local C^*-algebras. The algebras in the corona are called the quasi-local algebras. It is shown that these algebras are also $\pi_1(K)$-graded.

Ключевые слова: C^*-algebra, graded C^*-algebra, partially ordered set, net of C^*-algebras, net of Hilbert spaces, path semigroup, the first homotopy group, inductive limit.

The motivation for our work comes from papers [4, 5, 6] in which the nets of the C^*-algebras of observables are studied for a curved spacetime and a spacetime manifold with specific topological features.

We have studied earlier the C^*-algebras generated by representations of ordered semigroups [7, 8, 9, 10, 11, 12]. The present paper is a continuation of the study begun in the article [13]. There we dealt with the C^*-algebra generated by the path semigroup in a partially ordered set.

§2. Paths and loops on a partially ordered set

Let K be a partially ordered set with an order relation $\leqslant$, which is reflexive, antisymmetric and transitive. Elements a and b are said to be *comparable* in K, if $a \leqslant b$ or $b \leqslant a$. The set K is said to be *upward directed*, if for any $a, b \in K$ there exists $c \in K$ such that $a \leqslant c$ and $b \leqslant c$.

Further, we define paths on K. Ordered pairs (b, a) for $b \leqslant a$ and $\overline{(b, a)}$ for $b \geqslant a$ are called *elementary paths* on K. Here, the elements $\partial_1 p = a$ and $\partial_0 p = b$ are called, respectively, *the starting point* and *the ending point* of the elementary path. We define *the reverse elementary paths* $s^{-1} = \overline{(a, b)}$ for $s = (b, a)$ and $s^{-1} = (a, b)$ for $s = \overline{(b, a)}$. A pair $(a, a) = \overline{(a, a)} = i_a$ is called *a trivial path*.

Throughout we consider sequences of elementary paths of the following form:

$$\overline{p} = s_n * s_{n-1} * \ldots * s_1,$$

where $\partial_0 s_{i-1} = \partial_1 s_i$ for $i = 2, \ldots, n$. Here, the elements $\partial_1 \overline{p} = \partial_1 s_1$ and $\partial_0 \overline{p} = \partial_0 s_n$ are, respectively, the starting point and the ending point of the sequence $\overline{p}$. The reverse sequence defined as the sequence

$$\overline{p}^{-1} = s_1^{-1} * s_2^{-1} * \ldots * s_n^{-1}.$$

Extending the operation "$*$", we define the multiplication operation for the sequences of elementary paths $\overline{p} = s_n * \ldots * s_k$, and $\overline{q} = s_{k-1} * \ldots * s_1$ satisfying the condition $\partial_1 s_k = \partial_1 \overline{p} = \partial_0 \overline{q} = \partial_0 s_{k-1}$ as follows: $\overline{p} * \overline{q} = s_n * \ldots * s_k * s_{k-1} * \ldots * s_1$.

We denote by $\overline{S}$ the set of all sequences of elementary paths endowed with the operation "$*$". It is clear that the operation "$*$" is associative.

Let us define an equivalence relation on the set $\overline{S}$. To this end, for all elements $a, b, c \in K$ such that $a \leqslant b \leqslant c$, we put

$$(a, b) * (b, c) \sim (a, c); \tag{1}$$

$$\overline{(c, b)} * \overline{(b, a)} \sim \overline{(c, a)}; \tag{2}$$

$$(a, b) * \overline{(b, a)} \sim i_a; \tag{3}$$

$$\overline{(b, a)} * (a, b) \sim i_b. \tag{4}$$

It is worth noting that (1)–(4) imply the following equivalences:

$$(a, b) * i_b \sim (a, b);$$

$$i_a * (a, b) \sim (a, b);$$

$$\overline{(b, a)} * i_a \sim \overline{(b, a)};$$

$$i_b * \overline{(b, a)} \sim \overline{(b, a)};$$

$$i_a * i_a \sim i_a.$$

We put $\overline{p} \sim \overline{q}$, where $\overline{p}, \overline{q} \in \overline{S}$, if the sequence $\overline{p}$ can be obtained from the sequence $\overline{q}$ (and $\overline{q}$ from $\overline{p}$) by means of a finite number of relations (1)–(4).

One can easily verify that the following properties are fulfilled in $\overline{S}$:

1. for every sequence $\overline{p} \in \overline{S}$ with $\partial_0 \overline{p} = a$ and $\partial_1 \overline{p} = b$ the relations $\overline{p}^{-1} * \overline{p} \sim i_b$, $\overline{p} * \overline{p}^{-1} \sim i_a$ hold;

2. for every sequence $\overline{p} \in \overline{S}$, with $\partial_0 \overline{p} = a$ and $\partial_1 \overline{p} = b$ the relations $i_a * \overline{p} \sim \overline{p} \sim \overline{p} * i_b$ hold.

Let $p = [\overline{p}]$ be an equivalence class containing a sequence of elementary paths $\overline{p}$. For equivalence classes p and q we define the multiplication operation "$*$" as follows: if $\partial_1 \overline{p} = \partial_0 \overline{q}$ then we set

$$p * q = [\overline{p} * \overline{q}] = \{\overline{s} \sim \overline{p} * \overline{q} \mid \overline{s} \in \overline{S}\}.$$

Further, let us consider the quotient set of $\overline{S}$ by the equivalence relation. It is obvious that the quotient set $\overline{S}/_\sim$ is a groupoid. Adding a formal symbol 0 and putting

$$p * 0 = 0, \quad 0 * p = 0$$

for every $p \in \overline{S}/_\sim$, one may turn the groupoid $\overline{S}/_\sim$ into a semigroup denoted by

$$S = \overline{S}/_\sim \cup \{0\}.$$

Then for every $p, q \in S$ we have

$$p * q = \begin{cases} [\overline{p} * \overline{q}], & \text{if } p \neq 0,\ q \neq 0 \text{ and } \partial_1 \overline{p} = \partial_0 \overline{q}, \\ 0, & \text{otherwise.} \end{cases}$$

The semigroup S is called *the path semigroup*. An element $p \in S$ is called *a path* from the point $\partial_1 p = \partial_1 \overline{p}$ to the point $\partial_0 p = \partial_0 \overline{p}$. Thus, we identify all equivalent sequences of elementary paths. That equivalence class is called a path on K.

An element $\overline{p} \in \overline{S}$ is called *a loop in* $\overline{S}$ if the equality $\partial_0 \overline{p} = \partial_1 \overline{p}$ holds. For $a \in K$ we denote by $\overline{G}_a$ the set of all loops whose base point is a. The set $\overline{G}_a$ is a semigroup. For $a \in K$ we denote by G_a the set of all equivalence classes of loops whose base point is a. In [13] it is shown G_a is a subgroup in S with the unit $[i_a]$ and other properties of G_a and S are described. In particular, it is proved that if K is an upward directed set then G_a is trivial.

The set K is said to be *connected* provided that for every $a, b \in K$ there exists a path p such that $\partial_0 p = a$, $\partial_1 p = b$. It is shown in [13] that for a connected set K the isomorphism $G_a \cong G_b$ holds whenever $a, b \in K$. In particular, the mapping $\sigma_{ba} : G_a \to G_b$ defined by the formula

$$\sigma_{ba}(p) = [\overline{(b,a)} * \overline{p} * (a,b)] \tag{5}$$

is isomorphism, where $a \leqslant b$, $\overline{p} \in p$.

In what follows, we assume that the set K is connected.

The notion of *the first homotopy* (*fundamental*) *group* $\pi_1(K)$ for a partially ordered set K is given in [4, 6]. The group $\pi_1(K)$ is a quotient set of the set of all paths on K that start and end at the same point by the homotopy equivalence relation. Two paths are said to be *homotopy equivalent* if one can be obtained from the other by a finite number of elementary deformations (see [4, 6]).

Theorem 1. *For every $a \in K$ there exists an isomorphism $\pi_1(K) \cong G_a$.*

Proof. To prove the theorem it is sufficient to show that the equivalence relation given by formulas (1)–(4) coincides with the homotopy equivalence.

In [13] the authors show that if two paths are in the equivalence relation defined by (1)–(4) then they are homotopy equivalent.

For the converse implication we note that the elementary paths (a, b) and $\overline{(b, a)}$ are 1-simplices with the support b. Therefore every equivalence (1)–(4) is an elementary deformation of paths. $\square$

§3. Mappings and cycles on a Hilbert space

Let a net of Hilbert spaces

$$(K, H_a, \gamma_{ba})_{a \leqslant b \in K}$$

over K be given. Here, H_a is a Hilbert space with a basis $\{e_n^a\}_{n=1}^{\infty}$, and

$$\gamma_{ba} : H_a \to H_b$$

is an isometric embedding for $a \leqslant b$, which transforms the basis $\{e_n^a\}_{n=1}^\infty$ into the basis $\{e_n^b\}_{n=1}^\infty$ and satisfies the equation

$$\gamma_{ca} = \gamma_{cb} \circ \gamma_{ba},$$

whenever $a \leqslant b \leqslant c$. If $a = b$ then γ_{ba} is the identity mapping.

For every $a \in K$ we define the set $S_a = \{p \in S \mid \partial_0 p = a\}$.

Further, let us consider the Hilbert space of all square summable complex-valued functions on S_a

$$l^2(S_a) = \left\{ f : S_a \to \mathbb{C} \mid \sum_{p \in S_a} |f(p)|^2 < \infty \right\}$$

with the inner product given by

$$\langle f, g \rangle = \sum_{p \in S_a} f(p)\overline{g(p)}.$$

The family of functions $\{e_p\}_{p \in S_a}$ is an orthonormal basis in $l^2(S_a)$. Here, the equality $e_p(p') = \delta_{p,p'}$ holds for $p' \in S_a$, where $\delta_{p,p'}$ stands for the Kronecker symbol.

Let us consider the space $\mathcal{H} = \bigoplus_{a \in K}(H_a \otimes l^2(S_a))$. For every pair $a, b \in K$ satisfying the condition $a \leqslant b$ we define the partial isometry $\chi_a^b : \mathcal{H} \to \mathcal{H}$ as follows:

$$\chi_a^b(h \otimes e_p) = \begin{cases} \gamma_{ba}(h) \otimes e_{[\overline{(b,a)} * \overline{p}]}, & \text{if } h \in H_a \text{ and } \partial_0 p = a,\ \overline{p} \in p, \\ 0, & \text{otherwise.} \end{cases}$$

We note that the following inclusion holds:

$$\chi_a^b(H_a \otimes l^2(S_a)) \subseteq H_b \otimes l^2(S_b).$$

For the operator χ_a^b the conjugate operator $\chi_a^{b*} : \mathcal{H} \to \mathcal{H}$ is defined by the formula

$$\chi_a^{b*}(h \otimes e_p) = \begin{cases} h' \otimes e_{[(a,b) * \overline{p}]}, & \text{if there exists } h' \in H_a \text{ such that} \\ & \ h = \gamma_{ba}(h') \text{ and } \partial_0 p = b,\ \overline{p} \in p; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 1. *The following assertions hold.*

(1) $\chi_a^c = \chi_b^c \chi_a^b$ *whenever* $a \leqslant b \leqslant c$.

(2) $\chi_a^{c*} = \chi_a^{b*} \chi_b^{c*}$ *whenever* $a \leqslant b \leqslant c$.

(3) $\chi_a^{b*}\chi_a^b = I_{H_a} \otimes I_a$, *where* $I_{H_a} \otimes I_a : \mathcal{H} \to H_a \otimes l^2(S_a)$ *is a projection (a surjection).*

(4) $\chi_a^b \chi_a^{b*} = P_{H_b} \otimes I_b$, *where* $P_{H_b} \otimes I_b : \mathcal{H} \hookrightarrow H_b \otimes l^2(S_b)$ *is a projection (an injection).*

Proof. (1) It follows from the equality $\gamma_{ca} = \gamma_{cb} \circ \gamma_{ba}$ and the relation $\overline{(c,a)} * \overline{p} \sim \overline{(c,b)} * \overline{(b,a)} * \overline{p}$.

(2) Assume that the condition $\chi_a^{c*}(h \otimes e_p) = h'' \otimes e_{[(a,c)*\overline{p}]} \neq 0$ holds. Then we have $h = \gamma_{ca}(h'') = \gamma_{cb} \circ \gamma_{ba}(h'') = \gamma_{cb}(h')$. Consequently, we obtain the equalities

$$\chi_a^{b*}\chi_b^{c*}(h \otimes e_p) = \chi_a^{b*}(h' \otimes e_{[(b,c)*\overline{p}]}) = h'' \otimes e_{[(a,b)*(b,c)*\overline{p}]} = h'' \otimes e_{[(a,c)*\overline{p}]}.$$

(3) For every element $h \otimes e_p \in H_a \otimes l^2(S_a)$ one has the equalities $\chi_a^{b*}\chi_a^b(h \otimes e_p) = \chi_a^{b*}(\gamma_{ba}(h) \otimes e_{[\overline{(b,a)}*\overline{p}]}) = h \otimes e_{[(a,b)*\overline{(b,a)}*\overline{p}]} = h \otimes e_p$.

(4) Take an element $h \otimes e_p \in H_b \otimes l^2(S_b)$. If the property $\chi_a^b\chi_a^{b*}(h \otimes e_p) \neq 0$ is fulfilled then one can easily see that the equality $\chi_a^b\chi_a^{b*}(h \otimes e_p) = h \otimes e_p$ is valid. Hence, the inclusion $\chi_a^b\chi_a^{b*}(H_b \otimes l^2(S_b)) \subseteq H_b \otimes l^2(S_b)$ holds. □

The set of isometries $\{\chi_a^b\}_{a,b\in K, a\leqslant b}$ can be enlarged to the set $\{\chi_{\overline{p}}\}_{\overline{p}\in\overline{S}}$ as follows. Take an arbitrary sequence of elementary paths

$$\overline{p} = (a_{2n}, a_{2n-1})^{l_{2n-1}} * \ldots * (a_3, a_2)^{l_2} * (a_2, a_1)^{l_1},$$

where $l_k = 0, 1$ and $(a_{k+1}, a_k)^0 = (a_{k+1}, a_k)$, $(a_{k+1}, a_k)^1 = \overline{(a_{k+1}, a_k)}$, $k = 1, ..., 2n-1$. Then we have

$$\chi_{\overline{p}} = (\chi_{a_{2n}}^{a_{2n-1}})^{l_{2n-1}} \ldots (\chi_{a_2}^{a_3})^{l_2}(\chi_{a_2}^{a_1})^{l_1},$$

where $(\chi_{a_{k+1}}^{a_k})^0 = \chi_{a_{k+1}}^{a_k*}$, $(\chi_{a_{k+1}}^{a_k})^1 = \chi_{a_{k+1}}^{a_k}$, $k = 1, ..., 2n-1$.

We note that the equality $\chi_{\overline{p}}^* = \chi_{\overline{p}^{-1}}$ holds.

The set $\{\chi_{\overline{p}}\}_{\overline{p}\in\overline{S}}$ is closed with respect to the multiplication operation: $\chi_{\overline{p}}\chi_{\overline{q}} = \chi_{\overline{p}*\overline{q}}$ provided that the condition $\partial_1\overline{p} = \partial_0\overline{q}$ holds.

The set

$$H_{\overline{p}} = \{h \in H_{\partial_1\overline{p}} \mid \chi_{\overline{p}}(h \otimes e_q) \neq 0 \text{ if } \partial_0 q = \partial_1\overline{p}\}$$

is called *the domain* of the sequence $\overline{p}$.

Obviously, the set $H_{\overline{p}}$ is a Hilbert space. It is worth noting that, in general, we have the condition $H_{\overline{p}} \neq H_{\overline{q}}$ for two distinct sequences $\overline{p}, \overline{q} \in \overline{S}$, which are equivalent, i.e., $\overline{p} \sim \overline{q}$. For instance, $H_{\overline{p}} = H_a$ if $\overline{p} = (a,b) * \overline{(b,a)}$ with $a \leqslant b$, and $H_{\overline{q}} \subseteq H_a$ for $\overline{q} = \overline{(a,c)} * (c,a)$ with $c \leqslant a$. Here, the equivalences $\overline{p} \sim i_a \sim \overline{q}$ hold.

Thus, in general, one has the property $\chi_{\overline{p}} \neq \chi_{\overline{q}}$ for sequences satisfying the conditions $\overline{p} \sim \overline{q}$, $\overline{p} \neq \overline{q}$.

Lemma 2. *The following assertions hold*:

(1) *the operator* χ_{i_a} *is the identity mapping on* $H_a \otimes l^2(S_a)$;

(2) *if* $\overline{p} \sim \overline{q}$ *then* $\chi_{\overline{p}}(h \otimes e_s) = \chi_{\overline{q}}(h \otimes e_s)$ *for every* $h \in H_{\overline{p}} \cap H_{\overline{q}}$ *and* s *such that* $\partial_0 s = \partial_1\overline{p} = \partial_1\overline{q}$;

(3) *if* $\overline{p} \sim \overline{q}$ *and* $\gamma_{ba} : H_a \to H_b$ *is an isomorphism for all* $a \leqslant b \in K$, *then the equality* $\chi_{\overline{p}} = \chi_{\overline{q}}$ *holds.*

Proof. (1) It is obvious.

(2) It is enough to prove the assertion for equivalences (1)–(4). To this end, we assume that $a \leqslant b \leqslant c$. Then the equivalence $(a, b) * (b, c) \sim (a, c)$ holds. Take an element $h \otimes e_s$ such that

$$\chi_a^{b*}\chi_b^{c*}(h \otimes e_s) \neq 0 \quad \text{and} \quad \chi_a^{c*}(h \otimes e_s) \neq 0.$$

It follows from item 2 of Lemma 1 that one has the equality

$$\chi_a^{b*}\chi_b^{c*}(h \otimes e_s) = \chi_a^{c*}(h \otimes e_s).$$

Item 1 of Lemma 1 implies the assertion for the equivalence $\overline{(c, b)} * \overline{(b, a)} \sim \overline{(c, a)}$. Similarly, for equivalences (3) and (4) the assertion follows from items 3 and 4 of Lemma 1.

(3) It follows from (2) that equivalent deformations of the sequence $\overline{p}$ do not change values at points of the space $H_{\partial_1\overline{p}}$. Those deformations only restrict or extend the domain $H_{\overline{p}}$ of the sequence. Furthermore, if $\gamma_{ba} : H_a \to H_b$ is an isomorphism whenever $a \leqslant b \in K$, then we have the equalities $H_{\overline{p}} = H_{\overline{q}} = H_{\partial_1\overline{p}}$. Consequently, one gets the equality $\chi_{\overline{p}} = \chi_{\overline{q}}$, as required. $\square$

Theorem 2. *The mapping* $\pi : \overline{S} \to B(\mathcal{H})$ *given by* $\pi(\overline{p}) = \chi_{\overline{p}}$ *is a representation of* $(\overline{S}, *)$ *in the algebra of bounded operators* $B(\mathcal{H})$. *If each embedding* $\gamma_{ba} : H_a \to H_b$ *is an isomorphism for* $a \leqslant b \in K$, *then the mapping* π^* *defined by* $\pi^*([\overline{p}]) = \pi(\overline{p})$ *is a representation of the groupoid* $\overline{S}/_\sim$.

Proof. Take $\overline{p}, \overline{q} \in \overline{S}$ such that $\partial_1\overline{p} = \partial_0\overline{q}$. Then we have the equalities

$$\pi(\overline{p} * \overline{q}) = \chi_{\overline{p}*\overline{q}} = \chi_{\overline{p}}\chi_{\overline{q}} = \pi(\overline{p})\pi(\overline{q}).$$

Assume that all $\gamma_{ba} : H_a \to H_b$ are isomorphisms for all $a \leqslant b \in K$. Then, by Lemma 2, we have the equality $\chi_{\overline{p}} = \chi_{\overline{q}}$ for $\overline{p} \sim \overline{q}$. Hence, the mapping π^* in the statement of the theorem is well-defined. Moreover, it is a representation of the groupoid $\overline{S}/_\sim$. $\square$

In the sequel, if a sequence $\overline{p}$ is a loop, then the mapping $\chi_{\overline{p}}$ is called *a cycle.* We note that the equalities

$$\chi_{\overline{p}}\chi_{\overline{p}}^*\chi_{\overline{p}} = \chi_{\overline{p}}\,, \quad \chi_{\overline{p}}^*\chi_{\overline{p}}\chi_{\overline{p}}^* = \chi_{\overline{p}}^*$$

hold for every $\overline{p} \in \overline{G}_a$. Therefore the set of cycles $\{\chi_{\overline{p}}\}_{\overline{p}\in\overline{G}_a}$ is a regular semigroup. It is clear that the element $\chi_{\overline{p}}^*$ is unique for each cycle $\chi_{\overline{p}}$. As a consequence, the semigroup of cycles is inverse.

If the equivalence $\overline{p} \sim i_a$ holds for some $a \in K$, then the cycle $\chi_{\overline{p}}$ is said to be *trivial.*

Theorem 3. *Every trivial cycle $\chi_{\overline{p}}$ is a projection of the form*

$$\chi_{\overline{p}} = Q_{\overline{p}} \otimes I_a,$$

where $\overline{p} \in \overline{G}_a$ and $Q_{\overline{p}}$ is a projection on the domain $H_{\overline{p}}$.

Proof. By Lemma 2, since $\overline{p} \sim i_a$ for an element $a \in K$ the cycle $\chi_{\overline{p}}$ is a projection. □

Corollary 1. *For every loop $\overline{p}$ the equality $\chi^*_{\overline{p}}\chi_{\overline{p}} = Q_{\overline{p}} \otimes I_a$ holds, where $Q_{\overline{p}}$ is a projection on the domain $H_{\overline{p}^{-1}*\overline{p}} = H_{\overline{p}}$.*

Proof. We note that the equality $\chi^*_{\overline{p}}\chi_{\overline{p}} = \chi_{\overline{p}^{-1}*\overline{p}}$ and the equivalence $\overline{p}^{-1} * \overline{p} \sim i_a$ hold for an element $a \in K$. □

Corollary 2. *If $\overline{p}, \overline{q} \in \overline{G}_a$ and $\overline{p} \sim \overline{q}$, then the operator $\chi^*_{\overline{p}}\chi_{\overline{q}} = P_{H_a} \otimes I_a$ is a projection.*

Proof. It is sufficient to note that the following equivalences hold:

$$\overline{p}^{-1} * \overline{q} \sim \overline{p}^{-1} * \overline{p} \sim i_a.$$ □

Corollary 3. *For all trivial cycles $\chi_{\overline{p}}$ and $\chi_{\overline{q}}$ one has the equality*

$$\chi_{\overline{p}}\chi_{\overline{q}} = \chi_{\overline{q}}\chi_{\overline{p}}\,.$$

Theorem 4. *If K is an upward directed set then every cycle $\chi_{\overline{p}}$ is a projection of the form $\chi_{\overline{p}} = P_{H_a} \otimes I_a$, where $\overline{p} \in \overline{G}_a$ and P_{H_a} is a projection on the domain $H_{\overline{p}}$.*

Proof. In [13], it is shown that if K is an upward directed set then for each loop $\overline{p}$ one has the equivalence $\overline{p} \sim i_a$ for some $a \in K$. This means that every cycle $\chi_{\overline{p}}$ is trivial. Applying Theorem 3, we obtain the assertion of the theorem. □

A cycle $\chi_{\overline{p}}$ is said to be *finite* if the domain $H_{\overline{p}}$ is a finite-dimensional linear space.

A cycle $\chi_{\overline{p}}$ is said to be *nilpotent* if there exists a natural number m such that the equality $\chi^m_{\overline{p}} = 0$ holds.

§4. C^*-generated by cycles

In what follows we suppose that the set K is not upward directed.

In general case, for $\overline{p} \in \overline{G}_a$ every cycle $\chi_{\overline{p}}$ has the form $\chi_{\overline{p}} = U_{\overline{p}} \otimes T_{\overline{p}}$, where $U_{\overline{p}} : H_a \to H_a$ is a partial isometry and $T_{\overline{p}} : l^2(S_a) \to l^2(S_a)$ is a unitary operator corresponding to the loop $\overline{p}$ such that $T_{\overline{p}}e_q = e_{[\overline{p}*\overline{q}]}$, where $\overline{q} \in q$. By Theorem 3, if a cycle $\chi_{\overline{p}}$ is trivial then we may write the equality $\chi_{\overline{p}} = Q_{\overline{p}} \otimes I_a$, where $Q_{\overline{p}}$ is a projection.

Assume that we are given two equivalent loops $\overline{p} \sim \overline{q}$. Then one has the equality $T_{\overline{p}} = T_{\overline{q}}$, but, in general, we have $U_{\overline{p}} \neq U_{\overline{q}}$. Corollary 1 implies the equalities $\chi^*_{\overline{p}}\chi_{\overline{p}} = Q_{\overline{p}} \otimes I_a$ and $\chi^*_{\overline{q}}\chi_{\overline{q}} = Q_{\overline{q}} \otimes I_a$. For loops $\overline{p} \sim \overline{q}$ we define the order relation on cycles: $\chi_{\overline{p}} \leqslant \chi_{\overline{q}}$ if $Q_{\overline{p}} \leqslant Q_{\overline{q}}$. It is easy to verify that one has the relations

$$\chi^*_{\overline{p}}\chi_{\overline{q}} \leqslant \chi^*_{\overline{q}}\chi_{\overline{q}}\,; \quad \chi^*_{\overline{p}}\chi_{\overline{q}} \leqslant \chi^*_{\overline{p}}\chi_{\overline{p}}\,. \tag{6}$$

Indeed, to prove the first relation we rewrite it in the form $\chi_{\overline{p}^{-1}*\overline{q}} \leqslant \chi_{\overline{q}^{-1}*\overline{q}}$ and note that $Q_{\overline{p}^{-1}*\overline{q}} \leqslant Q_{\overline{q}} = Q_{\overline{q}^{-1}*\overline{q}}$. To prove the latter we make use of Corollary 2. This statement guarantees that the operator $\chi^*_{\overline{p}}\chi_{\overline{q}}$ is a projection. Hence, we obtain $\chi^*_{\overline{p}}\chi_{\overline{q}} = \chi^*_{\overline{q}}\chi_{\overline{p}} \leqslant \chi^*_{\overline{p}}\chi_{\overline{p}}$, as desired.

For loops $\overline{p} \sim \overline{q}$ we define the addition operation $\chi_{\overline{p}} \vee \chi_{\overline{q}}$ of cycles as follows:

1) if the operator $Q_{\overline{p}} + Q_{\overline{q}}$ is also projection, i.e., $Q_{\overline{p}}Q_{\overline{q}} = 0$, then we set $\chi_{\overline{p}} \vee \chi_{\overline{q}} = \chi_{\overline{p}} + \chi_{\overline{q}}$;

2) if the condition $Q_{\overline{p}}Q_{\overline{q}} = Q \neq 0$ holds then we put $\chi_{\overline{p}} \vee \chi_{\overline{q}} = \chi_{\overline{p}} + \chi_{\overline{q}}((Q_{\overline{q}} - Q) \otimes I_a)$.

Lemma 3. *Let $\overline{p} \sim \overline{q}$ be loops with base point a. Then the addition of cycles $\chi_{\overline{p}} \vee \chi_{\overline{q}}$ can be represented in the form $\chi_{\overline{p}} \vee \chi_{\overline{q}} = U_{\overline{p},\overline{q}} \otimes T_{\overline{p}} = U_{\overline{p},\overline{q}} \otimes T_{\overline{q}}$, where $U_{\overline{p},\overline{q}} : H_a \to H_a$ is a partial isometry.*

Proof. First, we assume that $Q_{\overline{p}}Q_{\overline{q}} = 0$. Since $T_{\overline{p}} = T_{\overline{q}}$ we get the equalities $\chi_{\overline{p}} \vee \chi_{\overline{q}} = \chi_{\overline{p}} + \chi_{\overline{q}} = (U_{\overline{p}} + U_{\overline{q}}) \otimes T_{\overline{p}}$, where $U_{\overline{p}} + U_{\overline{q}}$ is a partial isometry.

Second, we assume that $Q_{\overline{p}}Q_{\overline{q}} = Q \neq 0$. To prove the lemma it is enough to show that $(\chi^*_{\overline{p}} \vee \chi^*_{\overline{q}})(\chi_{\overline{p}} \vee \chi_{\overline{q}}) = \widehat{Q} \otimes I_a$, where $\widehat{Q}$ is a projection. Indeed, using relations (6), we have the following:

$$\begin{aligned}
(\chi_{\overline{p}} \vee \chi_{\overline{q}})^*(\chi_{\overline{p}} \vee \chi_{\overline{q}}) &= (\chi^*_{\overline{p}} + ((Q_{\overline{q}} - Q) \otimes I_a)\chi^*_{\overline{q}})(\chi_{\overline{p}} + \chi_{\overline{q}}((Q_{\overline{q}} - Q) \otimes I_a)) \\
&\leqslant Q_{\overline{p}} \otimes I_a + ((Q_{\overline{q}} - Q) \otimes I_a)(Q_{\overline{p}} \otimes I_a) \\
&\qquad + (Q_{\overline{p}} \otimes I_a)((Q_{\overline{q}} - Q) \otimes I_a) \\
&\qquad + ((Q_{\overline{q}} - Q) \otimes I_a)(Q_{\overline{q}} \otimes I_a)((Q_{\overline{q}} - Q) \otimes I_a) \\
&= Q_{\overline{p}} \otimes I_a + (Q_{\overline{q}} - Q) \otimes I_a = \widehat{Q} \otimes I_a.
\end{aligned}$$
□

Further, let E be an infinite subset in an equivalence class $[\overline{p}]$. We denote by $K(E)$ the family of all finite subsets of the set E. For every $A \in K(E)$ we define the operator

$$\chi_A = \bigvee_{\overline{q} \in A} \chi_{\overline{q}}.$$

It follows from Lemma 3 that χ_A is a partial isometry satisfying the property

$$\chi^*_A\chi_A = Q_A \otimes I_a,$$

where Q_A is a projection on the space

$$H_E = \bigcup_{\overline{p}\in E} H_{\overline{p}}.$$

As well as it was done for cycles one can define the order relation for all $A, B \in K(E)$ as follows:

$$\chi_A \leqslant \chi_B, \quad \text{if} Q_A \leqslant Q_B,$$

which is equivalent to the inclusion $A \subseteq B$.

Let χ_E be the limit with respect to the net $K(E)$ under the inclusion in the strong operator topology. In particular, if $E = [\overline{p}]$ then we get the operator $\chi_{[\overline{p}]} = \chi_p$. In the sequel we shall write

$$\chi_p = \bigvee_{\overline{p}\in p} \chi_{\overline{p}},$$

where the sum is taken over the entire equivalence class. We shall call this operator *the p-cycle.*

In the similar way as it was done for cycles, one can define a finite and a nilpotent p-cycles. In what follows, we suppose that every p-cycle χ_p is neither finite nor nilpotent. Although particular cycles $\chi_{\overline{p}}$ may be finite or nilpotent.

Lemma 4. *The following assertions hold*:

(1) *if* $\overline{p} \sim \overline{q}$ *then* $\chi_{[\overline{p}]} = \chi_{[\overline{q}]}$;

(2) *for every* $p \in G_a$ *the equalities* $\chi_p\chi_p^*\chi_p = \chi_p$ *and* $\chi_p^*\chi_p\chi_p^* = \chi_p^*$ *hold;*

(3) *for every* $p, q \in G_a$ *the relation* $\chi_p\chi_q \leqslant \chi_{p*q}$ *holds.*

Proof. (1) It follows immediately from the definition of the p-cycle.

(2) To prove the first equality we note that the representation

$$\chi_p^*\chi_p = Q_p \otimes I_a$$

holds, where Q_p is the projection on the space

$$H_p = \bigcup_{\overline{p}\in p} H_{\overline{p}}.$$

The proof of the second equation is similar.

(3) It is sufficient to show that the equality $\chi_p\chi_q = \chi_E$ holds for some $E \subseteq p * q$. Indeed, we have the equalities

$$\chi_p\chi_q = \bigvee_{\overline{p}\in p, \overline{q}\in q} \chi_{\overline{p}}\chi_{\overline{q}} = \bigvee_{\overline{p}\in p, \overline{q}\in q} \chi_{\overline{p}*\overline{q}}.$$

Then we get $E = \{\overline{p} * \overline{q} \mid \overline{p} \in p, \overline{q} \in q\} \subseteq \{\overline{s} \mid \overline{s} \in p * q\} = p * q$. This completes the proof. □

Further, we denote by $\mathfrak{A}_{a,e}$ the subalgebra in $B(\mathcal{H})$ generated by trivial cycles $\chi_{\overline{p}}$ with $\overline{p} \sim i_a$, which is closed in the strong operator topology. This algebra acts nontrivially only on the subspace $H_a \otimes l^2(S_a)$. We notice that this algebra is commutative and contains, in particular, the operators $\chi^*_{\overline{p}}\chi_{\overline{p}}$, $\chi^*_E\chi_E$ for $E \subseteq [\overline{p}]$, $\chi^*_p\chi_p$ and etc.

Let us consider the family of subspaces

$$\mathfrak{A}_{a,p} = \mathfrak{A}_{a,e}\chi_p\,, \quad p \in G_a.$$

The subalgebra $\mathfrak{A}_{a,e}$ corresponds to the unit $[i_a]$ of the group G_a. We claim that $\mathfrak{A}_{a,p}$ is a Banach space. Indeed, let us take a Cauchy sequence $\{A_n\chi_p\}_{n=1}^\infty$ in $\mathfrak{A}_{a,p}$. Hence, $\{A_n\chi_p\chi^*_p\}_{n=1}^\infty$ is a Cauchy sequence in $\mathfrak{A}_{a,e}$ as well. Since $\mathfrak{A}_{a,e}$ is a Banach space the sequence $\{A_n\chi_p\chi^*_p\}$ converges to some element $B \in \mathfrak{A}_{a,e}$. Then, by Lemma 4(2), we have $A_n\chi_p = A_n\chi_p\chi^*_p\chi_p$ and the sequence $\{A_n\chi_p\}$ converges to the element $B\chi_p \in \mathfrak{A}_{a,p}$, as claimed.

Let us denote by $\mathfrak{A}_a$ the subalgebra in $B(\mathcal{H})$ generated by elements from the family $\mathfrak{A}_{a,p}$, $p \in G_a$, which is closed with respect to the uniform norm.

The main result of this paragraph is the proof of the assertion stating that the C^*-algebra $\mathfrak{A}_a$ is a $\pi_1(K)$-graded algebra. For the definition of a G-graded C^*-algebra, where G is a group, we refer the reader to [14].

Lemma 5. *For every $E \subseteq [\overline{p}] = p$ we have $\chi_E \in \mathfrak{A}_{a,p}$. In particular, $\chi_{\overline{p}} \in \mathfrak{A}_{a,p}$ for each $\overline{p} \in p$.*

Proof. Assume that $E \subseteq p$. Then we have $\chi^*_E\chi_E \in \mathfrak{A}_{a,e}$ as well as $\chi^*_E\chi_E\chi_p \in \mathfrak{A}_{a,p}$. Further, one has the equality $\chi^*_E\chi_E = Q_E \otimes I_a$, where Q_E is a projection on the space

$$H_E = \bigcup_{\overline{p}\in E} H_{\overline{p}} \subseteq H_p.$$

Consequently, we have the equality $\chi^*_E\chi_E\chi_p = \chi_E$. □

Lemma 6. *For every $p, q \in G_a$ the inclusion $\chi_p\chi_q \in \mathfrak{A}_{a,p*q}$ holds.*

Proof. In the proof of Lemma 4 one has already seen that the equality

$$\chi_p\chi_q = \chi_E$$

holds for some $E \subseteq p * q$. Hence, by Lemma 5, we have the desired inclusion

$$\chi_p\chi_q \in \mathfrak{A}_{a,p*q}.$$ □

We recall that *a conditional expectation* is a positive linear operator Φ from a C^*-algebra $\mathfrak{A}$ to its subalgebra $\mathfrak{A}_0$ such that $\|\Phi\| = 1$ and $\Phi(BAC) = B\Phi(A)C$ for all $B, C \in \mathfrak{A}_0$ and $A \in \mathfrak{A}$.

Lemma 7. *The mapping $\Phi : \mathfrak{A}_a \to \mathfrak{A}_{a,e}$ given by $\Phi(B) = B_0$, where*

$$B = B_0 + \sum_k B_k \chi_{p_k} \in \bigoplus_{p \in G_a} \mathfrak{A}_{a,p}, \quad B_0, B_k \in \mathfrak{A}_{a,e}, \quad p_k \neq [i_a],$$

is a conditional expectation.

Proof. Firstly, let us show that for every B the inequality $\|\Phi(B)\| \leqslant \|B\|$ holds. It follows from the definition of the norm that for every $\varepsilon > 0$ there exists an element $h \otimes e_p \in H_a \otimes l^2(S_a)$ with $\|h \otimes e_p\| = 1$ such that we have the estimate

$$\|B_0(h \otimes e_p)\| \geqslant \|B_0\| - \varepsilon.$$

Then we obtain the following:

$$\|B\| \geqslant \|B(h \otimes e_p)\| = \Big\|B_0(h \otimes e_p) + \sum_k B_k \chi_{p_k}(h \otimes e_p)\Big\|$$

$$= \Big\|h_0 \otimes e_p + \sum_{k_i} h_{k_i} \otimes e_{p_{k_i} * p}\| \geqslant \|B_0(h \otimes e_p)\Big\| \geqslant \|B_0\| - \varepsilon.$$

The validity of the above-mentioned inequalities follows from the inclusion $\{p_{k_i}\} \subseteq \{p_k\}$ which guarantees the condition $p_{k_i} * p \neq p$. Since ε is arbitrary we get the required estimate

$$\|\Phi(B)\| = \|B_0\| \leqslant \|B\|.$$

Secondly, we take elements

$$B \in \bigoplus_{p \in G_a} \mathfrak{A}_{a,p}, \quad A, C \in \mathfrak{A}_{a,e}.$$

Then we have the equalities

$$\Phi(ABC) = \Phi(AB_0C + \sum_k AB_k \chi_{p_k} C)$$

$$= \Phi(B_0' + \sum_k AB_k C' \chi_{p_k}) = B_0' = AB_0C = A\Phi(B)C. \qquad \square$$

Theorem 5. *The C^*-algebra $\mathfrak{A}_a$ is $\pi_1(K)$-graded, that is, the following representation holds:*

$$\mathfrak{A}_a = \overline{\bigoplus_{p \in G_a \cong \pi_1(K)} \mathfrak{A}_{a,p}}\,.$$

Proof. It is obvious that $\mathfrak{A}_{a,p} \cap \mathfrak{A}_{a,q} = 0$ for $p \neq q$.

Let us show that the equality $\mathfrak{A}_{a,e}\chi_p = \chi_p \mathfrak{A}_{a,e}$ holds. Indeed, we take an element $P \otimes I_a \in \mathfrak{A}_{a,e}$. Using assertion (2) in Lemma 4 together with the commutativity of the algebra $\mathfrak{A}_{a,e}$, we obtain the following equalities:

$$(P \otimes I_a)\chi_p = (P \otimes I_a)\chi_p \chi_p^* \chi_p = \chi_p \chi_p^* (P \otimes I_a)\chi_p = \chi_p(Q \otimes I_a),$$

where

$$Q \otimes I_a = \chi_p^*(P \otimes I_a)\chi_p \in \mathfrak{A}_{a,e}.$$

This yields the desired equality.

By Lemma 6, for every $p, q \in G_a$ we obtain

$$\mathfrak{A}_{a,p}\mathfrak{A}_{a,q} \subseteq \mathfrak{A}_{a,e}\chi_p\mathfrak{A}_{a,e}\chi_q = \mathfrak{A}_{a,e}\chi_p\chi_q \subseteq \mathfrak{A}_{a,p*q}.$$

Further, for $P \otimes I_a \in \mathfrak{A}_{a,e}$ we have the equalities

$$((P \otimes I_a)\chi_p)^* = \chi_p^*(P \otimes I_a) = \chi_{p^{-1}}(P \otimes I_a) = (Q \otimes I_a)\chi_{p^{-1}}.$$

This means that the property $(\mathfrak{A}_{a,p})^* = \mathfrak{A}_{a,p^{-1}}$ is fulfilled.

Finally, Lemma 7 implies that there is no an element in the space $\mathfrak{A}_{a,p}$ that can be approximated by linear combinations of elements from the family $\{\mathfrak{A}_{a,q}\}_{q \in G_a \setminus \{p\}}$. □

§5. Corona for a net of C^*-algebras

The results of the preceding paragraph imply the existence of the family of the graded C^*-algebras $\{\mathfrak{A}_a\}_{a \in K}$ over the set K.

For elements $a \leqslant b \in K$ we define the mapping $\alpha_{ba} : \mathfrak{A}_a \to \mathfrak{A}_b$ as follows. Taking an element $\chi_{\overline{p}} \in \overline{G}_a$, we set

$$\alpha_{ba}(\chi_{\overline{p}}) = \chi_a^b \chi_{\overline{p}} \chi_a^{b*} = \chi_{\overline{(b,a)}*\overline{p}*(a,b)}.$$

If we have the equivalence $\overline{p} \sim i_a$ then we get the equivalence $\overline{(b,a)}*\overline{p}*(a,b) \sim i_b$ and the inclusion $\alpha_{ba}(\mathfrak{A}_{a,e}) \subseteq \mathfrak{A}_{b,e}$. Let $p \in G_a$. Then one has the equalities

$$\alpha_{ba}(\chi_p) = \chi_a^b \Big(\bigvee_{\overline{p} \in p} \chi_{\overline{p}}\Big) \chi_a^{b*} = \bigvee_{\overline{p} \in p} \chi_{\overline{(b,a)}*\overline{p}*(a,b)}.$$

Since the inclusion

$$\{\overline{(b,a)} * \overline{p} * (a,b) \mid \overline{p} \in p\} \subseteq [\overline{(b,a)} * \overline{p} * (a,b)] = \sigma_{ba}(p)$$

holds we conclude that

$$\alpha_{ba}(\chi_p) \in \mathfrak{A}_{b,\sigma_{ba}(p)},$$

where $\sigma_{ba} : G_a \to G_b$ is an isomorphism given by formula (5). Therefore we have the inclusion

$$\alpha_{ba}(\mathfrak{A}_{a,p}) \subseteq \mathfrak{A}_{b,\sigma_{ba}(p)}.$$

Thus, the mapping $\alpha_{ba} : \mathfrak{A}_a \to \mathfrak{A}_b$ preserves the graduation of the algebras. Moreover, this mapping is an embedding. Really, using Lemma 1, for all $A, B \in \mathfrak{A}_a$ we obtain the equalities

$$\alpha_{ba}(AB) = \chi_a^b AB \chi_a^{b*} = \chi_a^b A \chi_a^{b*} \chi_a^b B \chi_a^{b*} = \alpha_{ba}(A)\alpha_{ba}(B).$$

Lemma 1 implies that the property for the above mappings

$$\alpha_{ca} = \alpha_{cb} \circ \alpha_{ba}$$

is fulfilled whenever $a \leqslant b \leqslant c \in K$.

This means that the family of the algebras $\{\mathfrak{A}_a\}_{a\in K}$ constitutes the net of C^*-algebras

$$\big(K, \mathfrak{A}_a, \alpha_{ba}\big)_{a\leqslant b\in K}$$

over the set K, where each mapping $\alpha_{ba} : \mathfrak{A}_a \to \mathfrak{A}_b$ is an embedding. This net satisfies the isotony property (see [2]). The algebras of the net will be called *the local algebras*. We note that if all the mappings $\gamma_{ba} : H_a \to H_b$ are isomorphisms for $a \leqslant b \in K$ then the mappings $\alpha_{ba} : \mathfrak{A}_a \to \mathfrak{A}_b$ are isomorphisms of algebras.

We represent the partially ordered set K as the union of all its maximal upward directed subsets:

$$K = \bigcup_{i\in I} K_i.$$

Such a representation is unique. Further we consider the net

$$\big(K_i, \mathfrak{A}_a, \alpha_{ba}\big)_{a\leqslant b\in K_i}$$

over the upward directed set K_i. Since the mapping $\alpha_{ba} : \mathfrak{A}_a \hookrightarrow \mathfrak{A}_b$ is an embedding we may assume that the inclusion $\mathfrak{A}_a \subseteq \mathfrak{A}_b$ holds for $a \leqslant b$. We denote by

$$\mathfrak{A}_i = \overline{\bigcup_{a\in K_i} \mathfrak{A}_a}$$

the inductive limit of the system of the C^*-algebras $\{\mathfrak{A}_a\}_{a\in K_i}$ over the directed set K_i, that is, the completion with respect to the unique C^*-norm on $\bigcup_{a\in K_i} \mathfrak{A}_a$. The algebra $\mathfrak{A}_i$ is called *a quasi-local algebra.*

We call the family of the limit algebras $\{\mathfrak{A}_i\}_{i\in I}$ *the corona* for the net of C^*-algebras $(K, \mathfrak{A}_a, \alpha_{ba})_{a\leqslant b\in K}$.

Theorem 6. *In the corona for every $i \in I$ the algebra $\mathfrak{A}_i$ is a $\pi_1(K)$-graded C^*-algebra, that is, the following representation holds:*

$$\mathfrak{A}_i = \overline{\bigoplus_{p\in\pi_1(K)} \mathfrak{A}_{i,p}}.$$

Proof. It follows from the fact that the embedding $\alpha_{ba} : \mathfrak{A}_a \to \mathfrak{A}_b$ preserves the graduation of the algebras. We have the representations

$$\mathfrak{A}_{i,e} = \overline{\bigcup_{a\in K_i} \mathfrak{A}_{a,e}} \quad \text{and} \quad \mathfrak{A}_{i,p} = \overline{\bigcup_{a\in K_i} \mathfrak{A}_{a,p}}, \quad p \in \pi_1(K). \qquad \square$$

Assume we are given two nets

$$\big(K, H_a^K, \gamma_{ba}\big)_{a\leqslant b\in K} \quad \text{and} \quad \big(L, H_x^L, \gamma_{yx}\big)_{x\leqslant y\in L}$$

over partially ordered sets K and L, respectively, where H_a^K and H_x^L are Hilbert spaces and $\gamma_{ba} : H_a^K \to H_b^K$ as well as $\gamma_{yx} : H_x^L \to H_y^L$ are isometric embeddings for all $a \leqslant b$ and $x \leqslant y$.

A pair

$$(\varphi, \Phi) : \big(K, H_a^K, \gamma_{ba}\big)_{a\leqslant b\in K} \to \big(L, H_x^L, \gamma_{yx}\big)_{x\leqslant y\in L}$$

is called *a morphism for nets of Hilbert spaces* if the following properties are fulfilled:

1) $\varphi : K \to L$ is a morphism of partially ordered sets, i.e., the condition $a \leqslant b$ implies $\varphi(a) \leqslant \varphi(b)$;

2) the mapping

$$\Phi : \bigoplus_{a\in K} H_a^K \to \bigoplus_{x\in L} H_x^L$$

as well as the mappings

$$\Phi_a = \Phi|_{H_a^K} : H_a^K \hookrightarrow H_{\varphi(a)}^L$$

for all $a \in K$ are isometric embeddings;

3) the equality

$$\Phi_b \circ \gamma_{ba} = \gamma_{\varphi(b)\varphi(a)} \circ \Phi_a$$

holds whenever $a \leqslant b$.

Similarly, a pair

$$(\varphi, \Phi) : \big(K, \mathfrak{A}_a^K, \alpha_{ba}\big)_{a\leqslant b\in K} \to \big(L, \mathfrak{A}_x^L, \alpha_{yx}\big)_{x\leqslant y\in L}$$

is *a morphism for nets of C^*-algebras* if

$$\Phi = \{\Phi_a\}_{a\in K},$$

where $\Phi_a : \mathfrak{A}_a^K \to \mathfrak{A}_{\varphi(a)}^L$ is a $*$-homomorphism of C^*-algebras for every $a \in K$, and the equality

$$\Phi_b \circ \alpha_{ba} = \alpha_{\varphi(b)\varphi(a)} \circ \Phi_a$$

holds whenever $a \leqslant b$. A morphism is said to be *faithful* if Φ_a is an embedding for every $a \in K$.

Let $\{\mathfrak{A}_i^K\}_{i\in I}$ and $\{\mathfrak{A}_j^L\}_{j\in J}$ be the coronas for the nets of C^*-algebras $(K, \mathfrak{A}_a^K, \alpha_{ba})_{a\leqslant b\in K}$ and $(L, \mathfrak{A}_x^L, \alpha_{yx})_{x\leqslant y\in L}$, respectively. *A morphism of coronas* is a family of mappings $\Phi^* = \{\Phi_i^*\}_{i\in I}$ such that for every index $i \in I$ there exists an index $j \in J$ for which $\Phi_i^* : \mathfrak{A}_i^K \to \mathfrak{A}_j^L$ is a $*$-homomorphism of C^*-algebras.

A morphism $\varphi : K \to L$ induces the morphism $\overline{S}^K \to \overline{S}^L$, which is denoted by the same letter, as follows: if $\overline{p}$ is a sequence of elementary paths of the forms (a,b) and $\overline{(b,a)}$ on K then we set that $\varphi(\overline{p})$ is a similar sequence of elementary paths of the forms $(\varphi(a),\varphi(b))$ and $\overline{(\varphi(b),\varphi(a))}$ on L.

We notice that if $\overline{p}_1 \sim \overline{p}_2$ then $\varphi(\overline{p}_1) \sim \varphi(\overline{p}_2)$. Therefore, the morphism φ induces the homomorphisms of the groupoids

$$\varphi^* : \overline{S}^K/_\sim \to \overline{S}^L/_\sim$$

and groups

$$\varphi^* : G_a^K \to G_{\varphi(a)}^L$$

defined by $\varphi^*([\overline{p}]) = [\varphi(\overline{p})]$. Consequently, one gets the homomorphism of the first homotopy groups

$$\varphi^* : \pi_1(K) \to \pi_1(L).$$

Theorem 7. *Let $\varphi^* : \pi_1(K) \to \pi_1(L)$ be an injective morphism of the first homotopy groups and $\Phi_a : H_a^K \to H_{\varphi(a)}^L$ be an isometric isomorphism for every $a \in K$. Then the morphism for the nets of Hilbert spaces*

$$(\varphi,\Phi) : \left(K, H_a^K, \gamma_{ba}\right)_{a\leqslant b\in K} \to \left(L, H_x^L, \gamma_{yx}\right)_{x\leqslant y\in L}$$

induces the faithful morphism for nets of C^*-algebras

$$(\varphi,\Phi^*) : \left(K, \mathfrak{A}_a^K, \alpha_{ba}\right)_{a\leqslant b\in K} \to \left(L, \mathfrak{A}_x^L, \alpha_{yx}\right)_{x\leqslant y\in L}.$$

Proof. Let us consider the direct sums of Hilbert spaces

$$\mathcal{H}^K = \bigoplus_{a\in K} H_a^K \otimes l^2(S_a^K)$$

and

$$\mathcal{H}^L = \bigoplus_{x\in L} H_x^L \otimes l^2(S_x^L).$$

We define the mapping $\Phi \otimes \widehat{\varphi} : \mathcal{H}^K \to \mathcal{H}^L$ by setting

$$(\Phi \otimes \widehat{\varphi})(h \otimes e_p) = \Phi(h) \otimes e_{\varphi^*(p)}$$

for every $h \otimes e_p \in \mathcal{H}^K$. It is clear that the mapping

$$\Phi \otimes \widehat{\varphi} = \bigoplus_{a\in K} \Phi_a \otimes \widehat{\varphi}$$

is an isometric embedding and one has the inclusion

$$(\Phi_a \otimes \widehat{\varphi})(H_a^K \otimes l^2(S_a^K)) \subseteq H_{\varphi(a)}^L \otimes l^2(S_{\varphi(a)}^L).$$

We claim that for every $\overline{p} \in \overline{S}$ the equality

$$\widehat{\varphi} \circ T_{\overline{p}} = T_{\varphi(\overline{p})} \circ \widehat{\varphi}$$

holds, where $T_{\overline{p}}e_q = e_{[\overline{p}*\overline{q}]}$. Indeed, we have the equalities

$$\widehat{\varphi}T_{\overline{p}}e_q = \widehat{\varphi}e_{[\overline{p}*\overline{q}]} = e_{[\varphi(\overline{p})*\varphi(\overline{q})]} = T_{\varphi(\overline{p})}e_{\varphi^*(q)} = T_{\varphi(\overline{p})}\widehat{\varphi}e_q,$$

as claimed.

Let us show the validity of the equality

$$(\Phi_b \otimes \widehat{\varphi}) \circ \chi_a^b = \chi_{\varphi(a)}^{\varphi(b)} \circ (\Phi_a \otimes \widehat{\varphi}). \tag{7}$$

To this end, we write the operator χ_a^b in the form $\chi_a^b = \gamma_{ba} \otimes T_{\overline{(b,a)}}$. Then we have the chain of the following equalities:

$$\begin{aligned}(\Phi_b \otimes \widehat{\varphi})\chi_a^b &= (\Phi_b \otimes \widehat{\varphi})(\gamma_{ba} \otimes T_{\overline{(b,a)}}) \\ &= \Phi_b\gamma_{ba} \otimes \widehat{\varphi}T_{\overline{(b,a)}} = \gamma_{\varphi(b)\varphi(a)}\Phi_a \otimes T_{\overline{(\varphi(b),\varphi(a))}}\widehat{\varphi} \\ &= (\gamma_{\varphi(b)\varphi(a)} \otimes T_{\overline{(\varphi(b),\varphi(a))}})(\Phi_a \otimes \widehat{\varphi}) = \chi_{\varphi(a)}^{\varphi(b)}(\Phi_a \otimes \widehat{\varphi}).\end{aligned}$$

Since all mappings Φ_a are isometric isomorphisms we obtain the equality $\Phi_{\partial_1\overline{p}}(H_{\overline{p}}) = H_{\varphi(\overline{p})}$ for $\overline{p} \in \overline{S}$. Hence, it follows from (7) that

$$(\Phi_a \otimes \widehat{\varphi}) \circ \chi_a^{b*} = \chi_{\varphi(a)}^{\varphi(b)*} \circ (\Phi_b \otimes \widehat{\varphi}).$$

Therefore, for each $\overline{p}$ one has the equality

$$(\Phi_{\partial_0\overline{p}} \otimes \widehat{\varphi}) \circ \chi_{\overline{p}} = \chi_{\varphi(\overline{p})} \circ (\Phi_{\partial_1\overline{p}} \otimes \widehat{\varphi}).$$

We put $\Phi_a^*(\chi_{\overline{p}}) = \chi_{\varphi(\overline{p})}$ for every $\overline{p} \in \overline{G}_a$. If the equivalence $\overline{p} \sim i_a$ holds then we have the equivalence $\varphi(\overline{p}) \sim i_{\varphi(a)}$ as well. This means that if the cycle $\chi_{\overline{p}}$ is trivial in the algebra $\mathfrak{A}_a^K$ then the cycle $\chi_{\varphi(\overline{p})}$ is also trivial in the algebra $\mathfrak{A}_{\varphi(a)}^L$. Since Φ_a is an isomorphism and each trivial cycle has the form $\chi_{\overline{p}} = Q_{\overline{p}} \otimes I_a$ the mapping $\chi_{\overline{p}} \mapsto \chi_{\varphi(\overline{p})}$ defined on the set of generators can be extended to the embedding $\Phi_a^* : \mathfrak{A}_{a,e}^K \to \mathfrak{A}_{\varphi(a),e}^L$. Further, we extend the embedding Φ_a^* to the whole algebra $\mathfrak{A}_a^K$ as follows: if

$$\chi_p = \bigvee_{\overline{p}\in p} \chi_{\overline{p}}$$

then we set

$$\Phi_a^*(\chi_p) = \bigvee_{\overline{p}\in p} \Phi_a^*(\chi_{\overline{p}}) = \bigvee_{\overline{p}\in p} \chi_{\varphi(\overline{p})}.$$

Since the condition

$$\{\varphi(\overline{p}) \mid \overline{p} \in p\} \subseteq [\varphi(\overline{p})] = \varphi^*(p)$$

holds we get

$$\Phi_a^*(\chi_p) \in \mathfrak{A}_{\varphi(a),\varphi^*(p)}^L.$$

Thus, one has the inclusion $\Phi_a^*(\mathfrak{A}_{a,p}^K) \subseteq \mathfrak{A}_{\varphi(a),\varphi^*(p)}^L$. Moreover, because φ^* is an embedding we obtain the embedding

$$\Phi_a^* : \mathfrak{A}_a^K = \overline{\bigoplus_{p\in\pi_1(K)} \mathfrak{A}_{a,p}^K} \to \mathfrak{A}_{\varphi(a)}^L = \overline{\bigoplus_{g\in\pi_1(L)} \mathfrak{A}_{\varphi(a),g}^L}$$

that preserves the graduation.

It remains to check the equality

$$\Phi_b^* \circ \alpha_{ba} = \alpha_{\varphi(b)\varphi(a)} \circ \Phi_a^*$$

for every $a \leqslant b \in K$. To do this, we check its validity for the generators. Indeed, for $\overline{p} \in \overline{G}_a$ we have

$$\Phi_b^*\alpha_{ba}(\chi_{\overline{p}}) = \Phi_b^*(\chi_a^b \chi_{\overline{p}} \chi_a^{b*}) = \chi_{\varphi(a)}^{\varphi(b)} \chi_{\varphi(\overline{p})} \chi_{\varphi(a)}^{\varphi(b)*} = \alpha_{\varphi(b)\varphi(a)} \Phi_a^*(\chi_{\overline{p}}).$$

This completes the proof of the theorem. □

Corollary 4. *Let $\{\mathfrak{A}_i^K\}_{i\in I}$ and $\{\mathfrak{A}_j^L\}_{j\in J}$ be the coronas for nets*

$$\left(K, \mathfrak{A}_a^K, \alpha_{ba}\right)_{a\leqslant b\in K} \quad \text{and} \quad \left(L, \mathfrak{A}_x^L, \alpha_{yx}\right)_{x\leqslant y\in L},$$

respectively. Then a morphism for nets of C^*-algebras

$$(\varphi, \Phi^*) : \left(K, \mathfrak{A}_a^K, \alpha_{ba}\right)_{a\leqslant b\in K} \to (L, \mathfrak{A}_x^L, \alpha_{yx})_{x\leqslant y\in L}$$

is extended to a morphism of coronas $\Phi^* = \{\Phi_i^*\}_{i\in I}$ so that for every $i \in I$ there exists an index $j \in J$ such that

$$\Phi_i^*(\mathfrak{A}_i^K) \subseteq \mathfrak{A}_j^L.$$

Proof. Let us consider the inductive limit

$$\mathfrak{A}_i^K = \overline{\bigcup_{a\in K_i} \mathfrak{A}_a^K}.$$

We put

$$\Phi_i^*(\mathfrak{A}_i^K) = \overline{\bigcup_{a\in K_i} \Phi_a^*(\mathfrak{A}_a^K)} \subseteq \overline{\bigcup_{a\in K_i} \mathfrak{A}_{\varphi(a)}^L}.$$

Since the inclusion $\varphi(K_i) \subseteq L_j$ holds for some index $j \in J$ we have $\Phi_i^*(\mathfrak{A}_i^K) \subseteq \mathfrak{A}_j^L$, as required. □

The following example demonstrates that if φ^* is not an embedding then a morphism for nets of C^*-algebras is not faithful, in general.

Example. Let Y be the open unit disk in the complex plane with the center at the coordinate origin and $X = Y\backslash\{(0,0)\}$. Let K and L be the families of all open simply connected subsets of the sets X and Y, respectively. The families K and L are partially ordered sets under the inclusion relation. Moreover, the set L is upward directed. It is easy to see that the equalities $\pi_1(K) = \mathbb{Z}$ and $\pi_1(L) = \{0\}$ hold. The inclusion $X \subseteq Y$ yields the embedding $\varphi : K \to L$. Therefore, we have the homomorphism of the first homotopy groups $\varphi^* : \pi_1(K) \to \pi_1(L)$ such that $\varphi^*(n) = 0$ for each $n \in \mathbb{Z}$.

Further, let H be a Hilbert space. We consider the bundles of Hilbert spaces $(K, H_a, \gamma_{ba})_{a \leqslant b \in K}$ and $(L, H_x, \gamma_{yx})_{x \leqslant y \in L}$ over K and L, respectively (see [6]), where $H_a = H_x = H$ and γ_{ba}, γ_{yx} are the identity mappings. Then one associates to them the bundles of C^*-algebras $(K, \mathfrak{A}^K_a, \alpha_{ba})_{a \leqslant b \in K}$ and $(L, \mathfrak{A}^L_x, \alpha_{yx})_{x \leqslant y \in L}$ over K and L, respectively, where α_{ba} as well as α_{yx} are isomorphisms. The C^*-algebra $\mathfrak{A}^K_a$ is $\mathbb{Z}$-graded. It is generated by the operators $\chi_n = I \otimes T_0^n$, $n \in \pi_1(K)$. Here $I \otimes T_0$ is the unitary two-sided shift operator on the space $H \otimes l^2(S^K_a)$, which corresponds to $n = 1 \in \pi_1(K)$. Therefore, we have the isomorphism $\mathfrak{A}^K_a \cong C(S^1)$, where $C(S^1)$ is the Banach algebra of all continuous complex-valued functions on the unit circle in the complex plane. The C^*-algebra $\mathfrak{A}^L_x$ is generated by the operator $\chi_{\varphi^*(n)} = I \otimes I_a$. Hence, one has the isomorphism $\mathfrak{A}^L_x \cong \mathbb{C}$. The mapping $\Phi^*_a : \mathfrak{A}^K_a \to \mathfrak{A}^L_{\varphi(a)}$ defined by the correspondence $\chi_n \mapsto \chi_{\varphi^*(n)}$ yields the following commutative diagram:

$$\begin{tikzcd} \mathfrak{A}^K_a \arrow[r, "\Phi^*_a"] \arrow[d, "\cong"'] & \mathfrak{A}^L_{\varphi(a)} \arrow[d, "\cong"] \\ C(S^1) \arrow[r, "m"] & \mathbb{C} \end{tikzcd},$$

where $m : C(S^1) \to \mathbb{C}$ is the multiplicative functional given by

$$m(f) = f(1).$$

Obviously, the mapping m is not an embedding.

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Kazan State Power
Engineering University,
Krasnoselskaya st., 51
420066, Kazan, Russia
E-mail: gsuren@inbox.ru
E-mail: elipacheva@gmail.com
E-mail: airat_vm@rambler.ru

Поступило 1 декабря 2017 г.