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C^*-Algebra Generated by the Paths Semigroup

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Abstract—In this paper we study the structure of the C^*-algebra, generated by the representation of the paths semigroup on a partially ordered set (poset) and get the net of isomorphic C^*-algebras over this poset. We construct the extensions of this algebra, such that the algebra is an ideal in that extensions and quotient algebras are isomorphic to the Cuntz algebra.

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1. INTRODUCTION

In the algebraic approach to the quantum field theory [1] (the algebraic quantum field theory) the physical content of the theory is encoding by a collection of C^*-algebras of observables $\mathcal{A} = \{\mathcal{A}_o\}_{o\in K}$ indexed by elements of a partially ordered set K (poset) [2]. The poset K is a non-empty set with a binary relation $\leq$ which is reflexive, antisymmetric and transitive. A net of C^*-algebras over the poset K is the pair $(\mathcal{A}, \gamma)_K$, where $\gamma = \{\gamma_{o'o} : \mathcal{A}_o \to \mathcal{A}_{o'}\}_{o\leq o'}$ are *-morphisms fulfilling the net relations

$$\gamma_{o''o} = \gamma_{o''o'} \circ \gamma_{o'o}$$

for all $o \leq o' \leq o'' \in K$. If we consider the poset K as a category in which objects are elements of K and morphisms are arrows (o, o') for all $o \leq o' \in K$, then the net of C^*-algebras represents a covariant functor from a poset K to category of unital C^*-algebras with *-morphisms (see for example [3, 4]). More precisely we have a net of C^*-algebras for an upward directed poset and in the event of non-upward directed we obtain a precosheaf of C^*-algebras [5–7].

In this paper we give an algebraic notion of a path on a poset K which turns out to be relevant to the point of view on a path as a sequence of 1-simplices. We introduce the paths semigroup S on the given poset K and construct a new C^*-algebra $C^*_{red}(S)$ generated by the representation of S. We consider both an upward directed set K and non-upward directed. The present paper is addressed to detailed study of the paths semigroup S and the C^*-algebra $C^*_{red}(S)$. We construct the net of isomorphic C^*-algebras $\{\mathcal{A}_a, \gamma_{ba}, a \leq b\}_{a,b\in K}$ over the poset K, where $\mathcal{A}_a$ are restrictions of the algebra $C^*_{red}(S)$ on Hilbert subspaces and $\gamma_{ba} : \mathcal{A}_a \to \mathcal{A}_b$ are *-isomorphisms, such that $\gamma_{cb} \circ \gamma_{ba} = \gamma_{ca}$ for all $a \leq b \leq c \in K$. In the last section we consider extensions $C^*_{red,n}(S)$ and $C^*_{red,\infty}(S)$ of the algebra $C^*_{red}(S)$. We prove that $C^*_{red}(S)$ is an ideal in $C^*_{red,n}(S)$ and also in $C^*_{red,\infty}(S)$. We show that quotient algebras $C^*_{red,n}(S)/C^*_{red}(S)$ and $C^*_{red,\infty}(S)/C^*_{red}(S)$ are isomorphic to the Cuntz algebra.

Several works in recent years have addressed the C^*-algebras generated by the left regular representations of a semigroups with reduction [8] and by the representations of an inverse semigroup [9–11]. In the paper [12] have shown that the Cuntz algebra can be represented as a C^*-crossed product by endomorphisms of the CAR algebra.

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2. PATHS SEMIGROUP

In this section we define the paths semigroup S on a partially ordered set K. The semigroup S is an inverse semigroup and has subgroups G_a corresponding to loops which start and end at the same point $a \in K$.

Let K be a partially ordered set with binary relation $\leq$ satisfying reflexivity, antisymmetry and transitivity conditions. We call the set K a *poset*. Elements a and b are called *comparable* on K if $a \leq b$ or $b \leq a$. We say that the poset K is *upward directed* if for every pair $a, b \in K$ there exists $c \in K$, such that $a \leq c$ and $b \leq c$.

We call an ordered pair of comparable elements a and b on K an *elementary path*. We denote it by (b, a) if $b \leq a$ and by $\overline{(b, a)}$ if $b \geq a$ and say that a is a *starting point* of p, and b is an *ending point*. We use the denotation $\partial_1 p = a$ to denote the starting point of p and $\partial_0 p = b$ to denote the ending point. For an elementary path $p = (b, a)$ we define the *inverse path* $p^{-1} = \overline{(a, b)}$. For $p = \overline{(b, a)}$ the inverse path is $p^{-1} = (a, b)$. Obviously, $(p^{-1})^{-1} = p$. Finally we call the pair $(a, a) = \overline{(a, a)} = i_a$ a *trivial path*.

Let $p_1, \dots, p_n$ be elementary paths, such that $\partial_0 p_{i-1} = \partial_1 p_i$ for $i = 2, \dots, n$. We define a *path* p as the sequence

$$p = p_n * p_{n-1} * \dots * p_1.$$

The starting point of p is $\partial_1 p = \partial_1 p_1$ and the ending point is $\partial_0 p = \partial_0 p_n$. For every path $p = p_n * p_{n-1} * \dots * p_1$ the inverse path is

$$p^{-1} = p_1^{-1} * p_2^{-1} * \dots * p_n^{-1}$$

with $\partial_1 p^{-1} = \partial_0 p$ and $\partial_0 p^{-1} = \partial_1 p$. Let us consider a set of all paths on K. We define a semigroup structure on this set by extending the operation "$*$" to multiplication as

$$p * q = \begin{cases} p * q & \text{if } \partial_1 p = \partial_0 q, \\ 0 & \text{otherwise} \end{cases}$$

for all paths p and q.

The poset K is called *connected* if for all $a, b \in K$ there exists a path p, such that $\partial_0 p = a$, $\partial_1 p = b$. Throughout the rest of this article we assume K be a connected set.

We call the set of all paths on K a *paths semigroup S* if for all $a, b, c \in K$, such that $a \leq b \leq c$, the following axioms hold:

1. $(a, b) * (b, c) = (a, c)$;
2. $\overline{(c, b)} * \overline{(b, a)} = \overline{(c, a)}$;
3. $\overline{(b, a)} * (a, b) = i_b$, $(a, b) * \overline{(b, a)} = i_a$;
4. $(a, b) * i_b = (a, b)$, $i_a * (a, b) = (a, b)$;
5. $\overline{(b, a)} * i_a = \overline{(b, a)}$, $i_b * \overline{(b, a)} = \overline{(b, a)}$;
6. $i_a * i_a = i_a$.

It is easy to see that paths semigroup S has the following useful properties:

1) for every $p \in S$, such that $\partial_0 p = a$, $\partial_1 p = b$,

$$p^{-1} * p = i_b, \quad p * p^{-1} = i_a;$$

2) for every $p \in S$, such that $\partial_0 p = a$, $\partial_1 p = b$,

$$i_a * p = p * i_b = p;$$

3) for all $p, q \in S$, such that $\partial_0 q = \partial_1 p$,

$$(p * q)^{-1} = q^{-1} * p^{-1};$$

4) for all $p, q, s \in S$ if $p * q = p * s \neq 0$ or $q * p = s * p \neq 0$ then $q = s$; so the paths semigroup S is a semigroup with a reduction.

Thus, we can write elements of S as follows:

$$p = (a_{2n}, a_{2n-1}) * \dots * \overline{(a_3, a_2)} * (a_2, a_1) * \overline{(a_1, a_0)}. \tag{1}$$

Here elementary paths of type (a,b) and $\overline{(a,b)}$ alternate with each other. Note that there exists a variety of representations of type (1) for a path p. Our definition of the path turns out to be in full accordance with the definition given in [4]. The multiplication $(a_{i+1},a_i)*\overline{(a_i,a_{i-1})}$ is 1-*simplex* with support a_i where elements a_{i-1},a_i,a_{i+1} are 0-*simplices* (see definitions of 0-simplex and 1-simplex in [2–4]).

A three elements $a,c,x\in K$, such that $a,c\le x$, form 1-simplex denoted by

$$[a^x c]=(a,x)*\overline{(x,c)}$$

with support x. An inverse 1-simplex is

$$[c^x a]=(c,x)*\overline{(x,a)}$$

with the same support. In general 1-simplex depends on the support. But for example if $x,y\in K$ are comparable elements then

$$[a^x c]=[a^y c]. \tag{2}$$

Indeed for $x\le y$ we observe

$$[a^y c]=(a,y)*\overline{(y,c)}=(a,x)*(x,y)*\overline{(y,x)}*\overline{(x,c)}=(a,x)*i_x*\overline{(x,c)}=[a^x c].$$

In Lemma 4 we show that 1-simplex does not depend from the support if the poset is upward directed.

Therefore, one can rewrite the path (1) as a sequence of 1-simplices:

$$p=[{a_{2n}}^{a_{2n-1}}a_{2n-2}]*\ldots*[{a_2}^{a_1}a_0].$$

Let us recall the definition of an inverse semigroup (for details see [13–15]). Let S be a semigroup. Elements $a,b\in S$ are called *mutual inverses* if

$$a=aba,\quad b=bab.$$

The semigroup S is called an *inverse semigroup* if for every $a\in S$ there exists a unique inverse element $b\in S$.

We use the following theorem in the proof of Lemma 1.

Theorem 1 ([15]). *For a semigroup S in which every element has an inverse, uniqueness of inverses is equivalent to the requirement that all idempotents in S commute.*

Lemma 1. *The paths semigroup S is an inverse semigroup.*

Proof. Let $p\in S$ be a path with a starting point $\partial_1 p=a$ and an ending point $\partial_0 p=b$. For every p there is an inverse path p^{-1}, such that

$$p*p^{-1}*p=i_b*p=p,\quad p^{-1}*p*p^{-1}=i_a*p^{-1}=p^{-1}.$$

Hence, p and p^{-1} are mutual inverses elements. For every $a\in K$ we have $i_a*i_a=i_a$ and $i_a*i_b=0$ for all $a\ne b$. Therefore the set $\{i_a\}_{a\in K}$ forms a commutative subsemigroup of idempotents in the paths semigroup S. Hence, by Theorem 1 the paths semigroup S is an inverse semigroup. □

Lemma 2. *If for some* 1-simplices $[a^x b]$ *and* $[b^y c]$ *there exists* $z\in K$, *such that* $x,y\le z$, *then* $[a^x b]*[b^y c]=[a^z c]$.

Proof. We have

$$\begin{aligned}[a^x b]*[b^y c]&=(a,x)*\overline{(x,b)}*(b,y)*\overline{(y,c)}\\&=(a,x)*(x,z)*\overline{(z,x)}*\overline{(x,b)}*(b,y)*(y,z)*\overline{(z,y)}*\overline{(y,c)}\\&=(a,z)*\overline{(z,b)}*(b,z)*\overline{(z,c)}=(a,z)*\overline{(z,c)}=[a^z c].\end{aligned}$$

□

Corollary 1. *If for some 1-simplices* $[a^x b]$, $[b^y c]$ and $[a^z c]$ *there exists* $w\in K$, *such that* $x,y,z\le w$, *then* $[a^x b]*[b^y c]=[a^z c]$.

Proof. Using the Lemma 2 and the equality (2) we have $[a^x b]*[b^y c]=[a^w c]=[a^z c]$. □

In the works [3, 4] there exists the notion of an elementary deformation of a path. They say that a path admits an *elementary deformation* if one can replace some section $[a^x b]*[b^y c]$ of the path with $[a^z c]$ and vice versa. It is possible in the conditions of the Corollary 1. If we can obtain a path $q\in S$ from some path $p \in S$ by a finite number of elementary deformations then according to the Lemma 2 and the Corollary 1 we have the equality $q = p$.

We say that $p \in S$ is a *loop* if $\partial_0 p = \partial_1 p$.

Let us denote by G_a the set of all loops that start and end in the point a.

Lemma 3. *The following statements hold:*

1) *the set G_a is a subgroup in S with a unit i_a;*

2) *each path p generates isomorphism between groups G_a and G_b if $\partial_0 p = a, \quad \partial_1 p = b$;*

3) *if $p, q \in S$ and $\partial_0 p = \partial_0 q = a, \quad \partial_1 p = \partial_1 q = b$, then there exist $g_1 \in G_a$ and $g_2 \in G_b$, such that $p = g_1 * q = q * g_2$.*

Proof. 1) The first statement is obvious.

2) Define a map $\gamma_p : G_a \to G_b$ in the following way:

$$\gamma_p(g) = p^{-1} g p,$$

where $g \in G_a$. One can check that γ_p is an isomorphism.

3) It is easy to see that the statement holds for $g_1 = p * q^{-1} \in G_a$ and $g_2 = q^{-1} * p \in G_b$. □

Lemma 4. *If the poset K is an upward directed set then the following statements hold:*

1) *for all $a, b, x, y \in K$ if $a, b \leq x$ and $a, b \leq y$ then*

$$[a^x b] = (a, x) * \overline{(x, b)} = (a, y) * \overline{(y, b)} = [a^y b];$$

for simplicity let us omit supports and denote 1-simplex by $[a, b]$;

2) $[a, b] * [b, c] = [a, c]$ *for all $a, b, c \in K$;*

3) *for every $p \in S$ if $\partial_0 p = a$ and $\partial_1 p = b$* then $p = [a, b]$;

4) *if $g \in G_a$ then $g = i_a$ and the group G_a* is a trivial group.

Proof. 1) As the poset K is upward directed set then there exists $z \in K$, such that $x, y \leq z$. Hence, we have

$$\begin{aligned}[a^x b] &= (a, x) * \overline{(x, b)} = (a, x) * (x, z) * \overline{(z, x)} * \overline{(x, b)} = (a, z) * \overline{(z, b)} \\ &= (a, y) * (y, z) * \overline{(z, y)} * \overline{(y, b)} = (a, y) * \overline{(y, b)} = [a^y b].\end{aligned}$$

2) It follows from Lemma 2.

3) It follows from 2).

4) For every $g \in G_a$ we have $g = [a, a_n] * \ldots * [a_2, a_1] * [a_1, a]$. Using 2) several times, one gets $g = [a, a_1] * [a_1, a] = [a, a] = (a, a) * \overline{(a, a)} = i_a$. □

3. C^*-ALGEBRA $C^*_{red}(S)$

In this section we define the C^*-algebra $C^*_{red}(S)$ generated by the representation of the paths semigroup S and obtain the net of isomorphic C^*-algebras $(\mathcal{A}_a, \gamma_{ba}, a \leq b)_{a,b \in K}$ over the poset K, where $\gamma_{ba} : \mathcal{A}_a \to \mathcal{A}_b$ are *-isomorphisms satisfying the identity $\gamma_{cb} \circ \gamma_{ba} = \gamma_{ca}$ for $a \leq b \leq c$.

Let us consider a Hilbert space

$$l^2(S) = \left\{ f : S \to \mathbb{C} \,\middle|\, \sum_{p \in S} |f(p)|^2 < \infty \right\}$$

with an inner product $\langle f, g \rangle = \sum_{p \in S} f(p) \overline{g(p)}$. A family of functions $\{e_p\}_{p \in S}$ is an ortonormal basis of $l^2(S)$ where $e_p(p') = \delta_{p,p'}$ is a Kronecker symbol. Let $B(l^2(S))$ be the algebra of all linear bounded operators acting on $l^2(S)$.

Define a representation $\pi : S \to B(l^2(S))$ by $\pi(p) = T_p$ where

$$T_p e_q = \begin{cases} e_{p*q} & \text{if } \partial_1 p = \partial_0 q, \\ 0 & \text{otherwise.} \end{cases}$$

Note that π is the left regular representation and coincides with the Vagner representation of an inverse semigroup (see the definition of the Vagner representation in [14]).

We have $\langle T_p e_q, e_r\rangle \neq 0$ if and only if $p * q = r$ or $q = p^{-1} * r$. Hence,

$$\langle T_p e_q, e_r\rangle = \left\langle e_q, T_{p^{-1}} e_r\right\rangle .$$

Define the adjoint operator $T_p^* = T_{p^{-1}}$. In Lemma 5 we show that operators T_p and T_p^* are partial isometric operators.

Given $a \in K$ we define $S_a = \{p \in S | \partial_0 p = a\}$. Thus $l^2(S)$ can be written as

$$l^2(S) = \underset{a\in K}{\oplus} l^2(S_a).$$

Lemma 5. *The following statements hold:*

1) *for every $p \in S$, such that $\partial_0 p = a$, $\partial_1 p = b$, the operator T_p is a mapping from $l^2(S_b)$ to $l^2(S_a)$ and the operator T_p^* is an inverse mapping from $l^2(S_a)$ to $l^2(S_b)$;*

2) *for every $p \in S$, such that $\partial_0 p = a$, $\partial_1 p = b$, operators $I_a = T_p T_p^*$ and $I_b = T_p^* T_p$ are projectors on $l^2(S_a)$ and $l^2(S_b)$ respectively;*

3) *for every $g \in G_a$ the operator T_g is a unitary operator on $l^2(S_a)$;*

4) *for all $p, q \in S$, such that $\partial_0 p = \partial_0 q = a$, $\partial_1 p = \partial_1 q = b$, there exist $g_1 \in G_a$ and $g_2 \in G_b$, such that $T_p = T_{g_1} T_q = T_q T_{g_2}$.*

Proof. 1) We observe that $T_p e_q = e_{p*q}$ if $\partial_0 q = b$ and $T_p e_q = 0$ otherwise. Since $\partial_0(p * q) = a$ then $T_p : l^2(S_b) \to l^2(S_a)$. Similarly, $T_p^* : l^2(S_a) \to l^2(S_b)$.

2) It is easy to see that $I_a e_q = T_p T_p^* e_q = e_{p*p^{-1}*q} = e_q$ if $\partial_0 q = a$ and $I_a e_q = 0$ otherwise. Therefore, I_a is a projector on $l^2(S_a)$. Similarly, one can prove that I_b is a projector on $l^2(S_b)$.

3) We have $T_g : l^2(S_a) \to l^2(S_a)$ and $T_g T_g^* e_p = e_{g*g^{-1}*p} = e_p$, $T_g^* T_g e_p = e_p$ for every $p \in S_a$. Hence, T_g is a unitary operator.

4) This statement follows from the Lemma 3 (item 3). □

Let us denote by $C^*_{red}(S)$ a uniformly closed subalgebra of $B(l^2(S))$ generated by operators T_p for every $p \in S$. Obviously the set of finite linear combinations of operators T_p, $p \in S$, is dense in т $C^*_{red}(S)$.

Given $a \in K$ we denote $S^a = \{p \in S | \partial_1 p = a\}$. Thus we have again

$$l^2(S) = \underset{a\in K}{\oplus} l^2(S^a).$$

Theorem 2. *The following statements hold:*

1) *the algebra $C^*_{red}(S)$ is irreducible on $l^2(S^a)$ for every $a \in K$;*

2) *$C^*_{red}(S) = \underset{a\in K}{\oplus} C^*_{red}(S)|_{l^2(S^a)}$ and every operator $A \in C^*_{red}(S)$ can be represented as $A = \underset{a\in K}{\oplus} A_a$ where $A_a = A|_{l^2(S^a)}$;*

3) *if the group G_a is non-trivial then $C^*_{red}(S)|_{l^2(S^a)}$ doesn't contain compact operators.*

Proof. 1) The set $\{e_p, \partial_1 p = a\}_{p\in S}$ is a basis of $l^2(S^a)$. For all $p_1, p_2 \in S^a$ and $p = p_2 * p_1^{-1}$ we have $T_p e_{p_1} = e_{p_2}$. It means that the algebra $C^*_{red}(S)$ is irreducible on $l^2(S^a)$.

2) This statement follows from the fact that for every $p \in S$ operator T_p maps the space $l^2(S^a)$ onto itself for every $a \in K$.

3) Let $p \in S^a$, $g \in G_a$ and $g \neq i_a$. Consider the sequence $x_n = e_{p*g^n}$ where $g^n = \underbrace{g * g * \ldots * g}_{n}$. Since $g * g \neq g$ elements of the sequence $\{x_n\}$ are pairwise orthogonal. If $A \in C^*_{red}(S)|_{l^2(S^a)}$ is a compact operator then $||Ax_n|| \to 0$ as $n \to \infty$. On the other hand $Ae_p = \sum_i \alpha_i e_{p_i}$ where $p_i \in S^a$ and α_i are complex coefficients. Referral to the fact that A is approximated by finite linear combinations of operators T_q, $q \in S$, and to the equality $T_q e_{p*g} = e_{q*p*g}$ we obtain $Ae_{p*g} = \sum_i \alpha_i e_{p_i*g}$. Similarly

$Ae_{p*g^n} = \sum_i \alpha_i e_{p_i*g^n}$ for all n. Therefore, for every n we have $||Ax_n|| = \left(\sum_i |\alpha_i|^2\right)^{1/2} > 0$. Hence, A is not a compact operator. $\square$

Theorem 3. *Let K be an upward directed set. Then the following statements hold:*

1) *for every $p \in S$, such that $\partial_0 p = a,\ \partial_1 p = b$, we have $T_p = T_{[a,b]}$;*

2) *for every $a \in K$ the algebra $C^*_{red}(S)|_{l^2(S^a)}$ coincides with the algebra of all compact operators on $B(l^2(S^a))$;*

3) *the algebra $C^*_{red}(S)$ is non-unital.*

Proof. 1) This statement follows from the Lemma 4.

2) The set $\{e_{[c,a]}\}_{c\in K}$ is a basis of $l^2(S^a)$. For every operator T_p we have $T_p e_{[c,a]} \neq 0$ if and only if $\partial_1 p = c$. Hence, $T_p = T_{[b,c]}$ for some b and $T_{[b,c]}e_{[c,a]} = e_{[b,a]}$. Therefore, $T_p|_{l^2(S^a)}$ is a one dimensional operator. So C^*-algebra $C^*_{red}(S)|_{l^2(S^a)}$ coincides with the algebra of all compact operators on $B(l^2(S^a))$.

3) By the Theorem 2 for every element $A \in C^*_{red}(S)$ we have $A = \underset{a\in K}{\oplus} A_a$ where $A_a \in C^*_{red}(S)|_{l^2(S^a)}$. If the algebra $C^*_{red}(S)$ has the unit I then $I_a = I|_{l^2(S^a)}$ is a compact operator in the infinite dimensional Hilbert space. This is a contradiction. $\square$

Given $a \in K$ we denote $\mathcal{A}_a = C^*_{red}(S)|_{l^2(S^a)}$.

Theorem 4. *There exists the set of *-isomorphisms $\{\gamma_{ba}, a \leq b\}_{a,b\in K}$:*

$$\gamma_{ba} : \mathcal{A}_a \to \mathcal{A}_b,$$

such that $\gamma_{cb} \circ \gamma_{ba} = \gamma_{ca}$ for all $a, b, c \in K$ and *$a \leq b \leq c$. And we obtain a net of isomorphic C^*-algebras $\{\mathcal{A}_a, \gamma_{ba}, a \leq b\}_{a,b\in K}$ over the poset K.*

Proof. Define a unitary operator $U_{ab} : l^2(S^a) \to l^2(S^b)$ for all $a, b \in K$, $a \leq b$, by

$$U_{ab}e_q = e_{q*(a,b)}$$

for every $q \in S^a$. Then $U^*_{ab} = U_{\overline{ba}} : l^2(S^b) \to l^2(S^a)$ is the adjoint operator. Obviously, $U^*_{ab}U_{ab} = id|_{l^2(S^a)}$ and $U_{ab}U^*_{ab} = id|_{l^2(S^b)}$. Let us define a mapping $\gamma_{ba} : \mathcal{A}_a \to \mathcal{A}_b$ by

$$\gamma_{ba}(A) = U_{ab}AU^*_{ab}$$

for every $A \in \mathcal{A}_a$. One can check that γ_{ba} is the *-isomorphism. It remains to check the equality $\gamma_{cb} \circ \gamma_{ba} = \gamma_{ca}$ for $a \leq b \leq c$. We observe that

$$(\gamma_{cb} \circ \gamma_{ba})(A) = \gamma_{cb}(\gamma_{ba}(A)) = U_{bc}U_{ab}AU^*_{ab}U^*_{bc}$$

for every $A \in \mathcal{A}_a$. Otherwise

$$U_{bc}U_{ab}e_q = U_{bc}e_{q*(a,b)} = e_{q*(a,b)*(b,c)} = e_{q*(a,c)} = U_{ac}e_q$$

for every $q \in S^a$ and similarly $U^*_{ab}U^*_{bc}e_p = U^*_{ac}e_p$ for every $p \in S^c$. So $(\gamma_{cb} \circ \gamma_{ba})(A) = \gamma_{ca}(A)$ for every $A \in \mathcal{A}_a$. $\square$

Remark . *The set of isomorphisms $\{\gamma_{ba}, a \leq b\}_{a,b\in K}$ can be extended from elementary paths to 1-simplices $\{\gamma_{[b^x a]}, a, b \leq x\}_{a,b,x\in K}$ by $\gamma_{[b^x a]} = \gamma^{-1}_{xb} \circ \gamma_{xa}$, so that they satisfy 1-cocycle identity* [4]: *$\gamma_{[c^y b]} \circ \gamma_{[b^x a]} = \gamma_{[c^z a]}$ for $[c^y b] * [b^x a] = [c^z a]$. Extending the set $\{\gamma_{[b^x a]}, a, b \leq x\}_{a,b,x\in K}$ to paths we get the set of isomorphisms $\{\gamma_p\}_{p\in S}$ satisfying the equality $\gamma_{p_2} \circ \gamma_{p_1} = \gamma_{p_2 * p_1}$* for all $p_1, p_2 \in S$ and $\partial_0 p_1 = \partial_1 p_2$.

4. EXTENSIONS OF THE C^*-ALGEBRA $C^*_{red}(S)$

In this section we consider the extensions of the algebra $C^*_{red}(S)$, such that this algebra is an ideal in that extensions and quotient algebras are isomorphic to the Cuntz algebra.

Let K be an upward directed countable set. By the lemma 4 for every path $p \in S$, such that $\partial_0 p = a,\ \partial_1 p = b$, we have $p = [a, b]$. Let us represent the set K as a finite union of countable disjoint sets

$$K = \bigcup_{i=1}^{n} E_i,$$

where $E_i \bigcap E_j = \emptyset$ for $i \neq j$.

We define one-to-one mappings $\phi_i : E_i \to K$, $i = 1, \ldots, n$, and operators $T_{\phi_i} : l^2(S) \to l^2(S)$ in the following way:

$$T_{\phi_i} = \bigoplus_{x \in E_i} T_{[x,\phi_i(x)]}, \quad i = 1, \ldots, n.$$

An adjoint operator of the operator T_{ϕ_i} is

$$T^*_{\phi_i} = \bigoplus_{x \in E_i} T^*_{[x,\phi_i(x)]} = \bigoplus_{x \in E_i} T_{[\phi_i(x),x]} = \bigoplus_{x \in K} T_{[x,\phi_i^{-1}(x)]}$$

.

The following equalities hold:

$$T^*_{\phi_i} T_{\phi_i} = id; \quad T^*_{\phi_i} T_{\phi_j} = 0, \quad i \neq j; \quad \sum_{i=1}^{n} T_{\phi_i} T^*_{\phi_i} = id.$$

Indeed every basis element has a form $e_{[a,b]}$. Therefore,

$$\begin{aligned} T^*_{\phi_i} T_{\phi_i} e_{[a,b]} &= T^*_{\phi_i} T_{[\phi_i^{-1}(a),a]} e_{[a,b]} = T^*_{\phi_i} e_{[\phi_i^{-1}(a),b]} \\ &= T_{[a,\phi_i^{-1}(a)]} e_{[\phi_i^{-1}(a),b]} = e_{[a,b]}. \end{aligned}$$

Analogously, since $E_i \bigcap E_j = \emptyset$ we have $T^*_{\phi_i} T_{\phi_j} e_{[a,b]} = 0$. Finally if $a \in E_k$ then

$$\begin{aligned} \left(\sum_{i=1}^{n} T_{\phi_i} T^*_{\phi_i} \right) e_{[a,b]} &= T_{\phi_k} T_{[\phi_k(a),a]} e_{[a,b]} = T_{\phi_k} e_{[\phi_k(a),b]} \\ &= T_{[a,\phi_k(a)]} e_{[\phi_k(a),b]} = e_{[a,b]}. \end{aligned}$$

Let us consider a uniformly closed subalgebra of $B(l^2(S))$ generated by operators $T_p, p \in S$, and T_{ϕ_i}, $i = 1, \ldots, n$. Denote it by $C^*_{red,n}(S)$. The algebra $C^*_{red,n}(S)$ is unital. Hence, it doesn't coincide with $C^*_{red}(S)$. It is an extension of algebra $C^*_{red}(S)$. Moreover the following lemma holds.

Lemma 6. *The algebra $C^*_{red}(S)$ is an ideal in $C^*_{red,n}(S)$.*

Proof. We have $T_{\phi_i} T_{[a,b]} = T_{[x,b]}$ for some $x \in K$ and $T_{[a,b]} T_{\phi_i} = T_{[a,y]}$ for some $y \in K$. Since every element $A \in C^*_{red}(S)$ can be approximated by finite linear combinations of operators $T_{[a,b]}$ then $T_{\phi_i} A$ and $A T_{\phi_i} \in C^*_{red}(S)$. □

Let us recall the definition of the Cuntz algebra. The *finite Cuntz algebra O_n* is a C^*-algebra generated by isometries $s_1, \ldots, s_n$ satisfying to the following conditions:

$$s_i^* s_j = \delta_{ij} id, \quad \sum_{i=1}^{n} s_i s_i^* = id.$$

The *infinite Cuntz algebra O_∞* is a C^*-algebra generated by $s_1, s_2, \ldots$ and relations

$$s_i^* s_j = \delta_{ij} id, \quad \sum_{i=1}^{n} s_i s_i^* \leq id$$

for every $n \in \mathbb{N}$.

Theorem 5. *There exist an isomorphism $C^*_{red,n}(S)/C^*_{red}(S) \cong O_n$ and a short exact sequence*

$$0 \to C^*_{red}(S) \xrightarrow{id} C^*_{red,n}(S) \xrightarrow{\pi} O_n \to 0,$$

where id is an embedding map and π is a quotient map.

Proof. Equivalence classes $[T_{\phi_i}] = T_{\phi_i} + C^*_{red}(S)$, $i = 1, \ldots, n$, are generators of the quotient algebra $C^*_{red,n}(S)/C^*_{red}(S)$. These classes are isometric operators satisfying the following identity:

$$\sum_{i=1}^{n} [T_{\phi_i}][T^*_{\phi_i}] = id.$$

Due to the universality of the Cuntz algebra we observe that

$$C^*_{red,n}(S)/C^*_{red}(S) \cong O_n.$$

□

Now let us represent the set K as a countable union of disjoint countable sets:

$$K = \bigcup_{i=1}^{\infty} E_i$$

and define operators $T_{\phi_i} : l^2(S) \to l^2(S)$ in the following way:

$$T_{\phi_i} = \bigoplus_{x \in E_i} T_{[x,\phi_i(x)]}, \quad i = 1, 2, \ldots.$$

By applying the reasoning used above one can prove the following equalities:

$$T^*_{\phi_i} T_{\phi_i} = id; \quad T^*_{\phi_i} T_{\phi_j} = 0, \quad i \neq j; \quad \sum_{i=1}^{n} T_{\phi_i} T^*_{\phi_i} \leq id$$

for every $n \in \mathbb{N}$.

Let us denote by $C^*_{red,\infty}(S)$ the uniformly closed subalgebra of $B(l^2(S))$ generated by operators T_p, $p \in S$, and T_{ϕ_i}, $i = 1, 2, \ldots$.

Similarly to the Lemma 6 the algebra $C^*_{red}(S)$ is an ideal in $C^*_{red,\infty}(S)$ and for the infinite Cuntz algebra the following theorem holds.

Theorem 6. *There exist an isomorphism $C^*_{red,\infty}(S)/C^*_{red}(S) \cong O_\infty$ and a short exact sequence*

$$0 \to C^*_{red}(S) \xrightarrow{id} C^*_{red,\infty}(S) \xrightarrow{\pi} O_\infty \to 0,$$

where id is an embedding map and π is a quotient map.

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