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A Semigroup C^*-algebra for a Semidirect Product of Semigroups

E. V. Lipacheva[1,2*]

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[1]*Kazan State Power Engineering University, Kazan, 420066 Russia*

[2]*Lobachevskii Institute of Mathematics and Mechanics, Kazan (Volga Region) Federal University, Kazan, 420008 Russia*

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Abstract—The paper deals with the reduced semigroup C^*-algebra for the semidirect product of semigroups S and P, where P acts on S by automorphisms. We represent this C^*-algebra as the reduced crossed product of the reduced semigroup C^*-algebra for S by the semigroup P which acts by automorphisms. The purpose of the paper is to demonstrate that the semicrossed product C^*-algebras and the semidirect products of semigroups are closely related. We show that the reduced semigroup C^*-algebra for a semidirect product $S \rtimes_\beta^a P$ is isomorphic to the reduced semicrossed product C^*-algebra $C_r^*(S) \rtimes_{\alpha,r}^a P$. We apply this result to the study of the structure of the reduced semigroup C^*-algebra for the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}^\times$ of the additive group $\mathbb{Z}$ of all integers and the multiplicative semigroup $\mathbb{Z}^\times$ of integers without zero.

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1. INTRODUCTION

In this paper, we study the reduced semigroup C^*-algebra for a semidirect product of semigroups S and P, where P acts on S by automorphisms. The main purpose of our work is to represent this C^*-algebra as the reduced semicrossed product of the reduced semigroup C^*-algebra $C_r^*(S)$ by P.

The reduced semigroup C^*-algebras are very natural objects. They are generated by the left regular representations of semigroups with the cancellation property. The start in studying these algebras was made by Coburn [1, 2] who considered the reduced semigroup C^*-algebra for the additive semigroup of the non-negative integers. Douglas [3] investigated the case of subsemigroups in the additive group of the real numbers. Murphy [4, 5] generalized the results from [1–3] to the case of the reduced semigroup C^*-algebras for the positive cones in ordered groups. For extensive literature and history of the study of semigroup C^*-algebras, the reader is referred, for example, to [6] and the references therein.

The subject of the crossed products of C^*-algebras by groups is a well-developed branch of the theory of C^*-algebras. On the one hand, the crossed products provide interesting examples of C^*-algebras. On the other hand, the problem on the representation of a C^*-algebra as a crossed product C^*-algebra attracts a great deal of attention because it has important applications to a variety of questions in the theory of C^*-algebras. A systematic exposition of the crossed products is contained in the monograph [7].

An important task of the modern research is the development of a similar theory for the crossed products of C^*-algebras by semigroups. A theory of the crossed products of C^*-algebras by semigroups of their automorphisms has been developed by Murphy [8]. For a certain class of semigroups, Laca and Raeburn [9] proposed the construction of the crossed product of a C^*-algebra by a semigroup of

*E-mail: elipacheva@gmail.com

endomorphisms. In the paper [10] this construction was generalized. In particular, there was identified the relationship between the crossed products by automorphisms and endomorphisms. In contemporary literature, the crossed product of C^*-algebras by semigroups are called the semicrossed product of C^*-algebras (see the survey [11]).

There are two types of the crossed products of a C^*-algebra $\mathcal{A}$ by a locally compact group G. Namely, these are the full and the reduced crossed products. The full crossed product $\mathcal{A} \rtimes_\alpha G$ should be thought as a twisted maximal tensor product of $\mathcal{A}$ with the full group C^*-algebra $C^*(G)$ of the group G. The reduced crossed product $\mathcal{A} \rtimes_{\alpha,r} G$ should be regarded as a twisted minimal (or spatial) tensor product of $\mathcal{A}$ by the reduced group C^*-algebra $C_r^*(G)$. We note that if the group G is amenable, then the full and the reduced crossed products coincide, that is, $\mathcal{A} \rtimes_\alpha G = \mathcal{A} \rtimes_{\alpha,r} G$.

In studying a semicrossed product of C^*-algebra by endomorphisms we have a serious drawback. Namely, we do not have an explicit nontrivial representation of the semicrossed product at hand. But, due to Murphy, there is a semicrossed product of C^*-algebra by automorphisms. And this time, we have a canonical nontrivial representation of the semicrossed product in analogy to the reduced crossed product by a group.

Our research was motivated by the relationship between the crossed products of algebras by groups and the semidirect products of groups. Suppose that H and G are locally compact groups and $\beta : G \longrightarrow \mathrm{Aut}(H)$ is a homomorphism such that the action $(g, h) \mapsto \beta_g(h)$ is continuous. Then, the semidirect product $H \rtimes_\beta G$ is a locally compact group. The action β of the group G can be extended from the group H to the C^*-algebra $C^*(H)$ (or $C_r^*(H)$). Denote this action by α. Then, there are the natural isomorphisms [12, II.10.3.15]

$$C^*(H \rtimes_\beta G) \cong C^*(H) \rtimes_\alpha G \quad \text{and} \quad C_r^*(H \rtimes_\beta G) \cong C_r^*(H) \rtimes_{\alpha,r} G. \tag{1}$$

As an example, consider the group algebra of the infinite dihedral group. This group is a generalization of the finite dihedral groups, which are the symmetry groups of the regular polygons. The infinite dihedral group can be interpreted as the symmetry group for the set of integers.

Note that the infinite dihedral group is the group $D_\infty := \mathbb{Z} \rtimes_\beta \mathbb{Z}_2$, where $\mathbb{Z}$ is the additive group of all integers, $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ is the cyclic group of order two and $\beta : \mathbb{Z}_2 \longrightarrow \mathrm{Aut}(\mathbb{Z})$ is the group homomorphism such that $\beta_0(n) = n$ and $\beta_1(n) = -n$ whenever $n \in \mathbb{Z}$.

We note that the group D_∞ is amenable (see, for example, [13, Section 1]). Therefore, the reduced group C^*-algebra $C_r^*(D_\infty)$ coincides with the full group C^*-algebra $C^*(D_\infty)$. So, using (1), we get the isomorphism of C^*-algebras

$$C_r^*(D_\infty) = C^*(D_\infty) = C^*(\mathbb{Z} \rtimes_\beta \mathbb{Z}_2) \cong C^*(\mathbb{Z}) \rtimes_\alpha \mathbb{Z}_2, \tag{2}$$

where $C^*(\mathbb{Z}) = C^*\{u | u^*u = uu^* = I\}$ is the universal C^*-algebra generated by a unitary element (see, for example, [14, Appendix]), and $\alpha_0 = id$, $\alpha_1(u) = u^*$.

In [15], we obtained an analogue of the second isomorphism in (1) for the reduced semigroup C^*-algebra of a discrete semigroup. Namely, let S be a discrete left cancelative semigroup, G be a discrete group and $\beta : G \longrightarrow \mathrm{Aut}(S)$ be a group homomorphism. Then, there exists an isomorphism

$$C_r^*(S \rtimes_\beta G) \cong C_r^*(S) \rtimes_{\alpha,r} G,$$

where $\alpha : G \longrightarrow \mathrm{Aut}(C_r^*(S))$ is the group homomorphism induced by the homomorphism β.

In the present paper, we get a similar result for the case when G is a semigroup. We consider discrete semigroups S and P with the left cancelation property and a semigroup homomorphism $\beta : P \longrightarrow \mathrm{Aut}(S)$. Then, the semidirect product $S \rtimes_\beta^a P$ is the left cancelative semigroup. In Section 2, we show that there exists an isomorphism

$$C_r^*(S \rtimes_\beta^a P) \cong C_r^*(S) \rtimes_{\alpha,r}^a P,$$

where $\alpha : P \longrightarrow \mathrm{Aut}(C_r^*(S))$ is the semigroup homomorphism induced by β. In Section 3, this result is applied to the reduced semigroup C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ which was studied in [16–18].

2. PRELIMINARIES

We begin by recalling the definition of the reduced semigroup C^*-algebra for a semigroup.

Let S be a discrete left cancelative semigroup. As usual, the symbol $l^2(S)$ stands for the Hilbert space of all square summable complex-valued functions on S. For every $a \in S$, we denote by e_a the function in $l^2(S)$ which is defined as follows: $e_a(b) = 1$, if $b = a$, and $e_a(b) = 0$, if $b \neq a$, $b \in S$. Then, the set of functions $\{e_a \mid a \in S\}$ is an orthonormal basis in the Hilbert space $l^2(S)$.

In the C^*-algebra of all bounded linear operators $B(l^2(S))$ on the Hilbert space $l^2(S)$, we define the C^*-subalgebra $C^*_r(S)$ generated by the set of isometries $\{T_a \mid a \in S\}$, where $T_a(e_b) = e_{ab}$ for $a, b \in S$. It is called *the reduced semigroup C^*-algebra*. The identity element in this algebra is denoted by I.

Now we recall the necessary notions concerning the semicrossed products by automorphisms. Such semicrossed products were considered by Murphy in [8, 19], then were developed by Li in [10, 20]. We define the full semicrossed product by automorphisms according to the paper [20, Appendix A].

Let $\mathcal{A}$ be a unital C^*-algebra, P be a discrete left cancelative semigroup and $\alpha : P \longrightarrow \mathrm{Aut}(\mathcal{A})$ be a semigroup homomorphism. The triple $(\mathcal{A}, P, \alpha)$ is called *a C^*-dynamical semisystem by automorphisms*.

A covariant representation of the C^*-dynamical semisystem $(\mathcal{A}, P, \alpha)$ is a triple $(\mathcal{D}, \pi, v)$ consisting of a unital C^*-algebra $\mathcal{D}$, a unital $*$-homomorphism $\pi : \mathcal{A} \longrightarrow \mathcal{D}$ and a homomorphism of semigroups $v : P \longrightarrow \mathrm{Isom}(\mathcal{D})$, where $\mathrm{Isom}(\mathcal{D})$ is the subsemigroup of isometric elements in $\mathcal{D}$, such that the covariance relation

$$\pi(\alpha_p(a))v(p) = v(p)\pi(a)$$

is fullfilled for all $a \in \mathcal{A}$ and $p \in P$. In what follows, we will write $v : P \longrightarrow \mathcal{D}$ instead of $v : P \longrightarrow \mathrm{Isom}(\mathcal{D})$.

A morphism $\Phi : (\mathcal{D}_1, \pi_1, v_1) \longrightarrow (\mathcal{D}_2, \pi_2, v_2)$ of two covariant representations of C^*-dynamical semisystem $(\mathcal{A}, P, \alpha)$ is a unital $*$-homomorphism $\phi : \mathcal{D}_1 \longrightarrow \mathcal{D}_2$ such that the following diagrams

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{D}_1 \arrow[rr, "\phi"] & & \mathcal{D}_2, \end{tikzcd} \qquad \begin{tikzcd} & P \arrow[dl, "v_1"'] \arrow[dr, "v_2"] & \\ \mathcal{D}_1 \arrow[rr, "\phi"] & & \mathcal{D}_2 \end{tikzcd}$$

are commutative, i.e., $\phi \circ \pi_1 = \pi_2$ and $\phi \circ v_1 = v_2$.

The (full) semicrossed product associated to the C^*-dynamical semisystem $(\mathcal{A}, P, \alpha)$ by automorphisms is the covariant representation $(\mathcal{A} \rtimes^a_\alpha P, j_\mathcal{A}, j_P)$ of $(\mathcal{A}, P, \alpha)$ satisfying the following universal property: for any covariant representation $(\mathcal{D}, \pi, v)$ of C^*-dynamical semisystem $(\mathcal{A}, P, \alpha)$ there exists a unique morphism of covariant representations

$$\Phi_{(\pi,v)} : (\mathcal{A} \rtimes^a_\alpha P, j_\mathcal{A}, j_P) \longrightarrow (\mathcal{D}, \pi, v).$$

Consider the category associated to the C^*-dynamical semisystem $(\mathcal{A}, P, \alpha)$. Objects of this category are covariant representations of $(\mathcal{A}, P, \alpha)$ and morphisms of this category are morphisms of covariant representations. Then, the semicrossed product associated to $(\mathcal{A}, P, \alpha)$ is the initial object of this category.

The term "semicrossed product" will always mean "full semicrossed product". The semicrossed product is unique up to an isomorphism. The C^*-algebra $\mathcal{A} \rtimes^a_\alpha P$ is called *the semicrossed product C^*-algebra*. The existence of the C^*-algebra $\mathcal{A} \rtimes^a_\alpha P$ is shown in [10].

It is worth noting that for every C^*-dynamical semisystem $(\mathcal{A}, P, \alpha)$ by automorphisms there exists a nontrivial semicrossed product. This is the difference between the semicrossed product by automorphisms and the semicrossed product by endomorphisms. The latter can be trivial in some bad cases. For details we refer the reader to [9, 20].

For every C^*-dynamical semisystem $(\mathcal{A}, P, \alpha)$ by automorphisms there exists a nontrivial covariant representation of $(\mathcal{A}, P, \alpha)$ in analogy to the left regular representation. Murphy has introduced this representation in [19]. So we can use Murphy's construction to define the reduced semicrossed product C^*-algebra.

Let $\pi : \mathcal{A} \longrightarrow B(H)$ be a faithful representation on a Hilbert space H. Let $H \otimes l^2(P)$ be the Hilbert tensor product. Define representations $\pi_\alpha : \mathcal{A} \longrightarrow B(H \otimes l^2(P))$ and $\lambda : P \longrightarrow B(H \otimes l^2(P))$ as follows

$$\pi_\alpha(a)(\xi \otimes e_h) = \pi(\alpha_h^{-1}(a))\xi \otimes e_h,$$
$$\lambda(p)(\xi \otimes e_h) = \xi \otimes e_{ph}, \tag{3}$$

where $a \in \mathcal{A}, p, h \in P, \xi \in H$ and the set of functions $\{e_h \mid h \in P\}$ is an orthonormal basis in the Hilbert space $l^2(P)$. It is easy to check that the triple $(B(H \otimes l^2(P)), \pi_\alpha, \lambda)$ is a covariant representation of the C^*-dynamical semisystem $(\mathcal{A}, P, \alpha)$.

Thus, by the universal property of $(\mathcal{A} \rtimes_\alpha^a P, j_\mathcal{A}, j_P)$ there exists a unique unital $*$-homomorphism $\phi_{(\pi_\alpha,\lambda)} : \mathcal{A} \rtimes_\alpha^a P \longrightarrow B(H \otimes l^2(P))$ such that the following diagrams

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "j_\mathcal{A}"'] \arrow[dr, "\pi_\alpha"] & \\ \mathcal{A} \rtimes_\alpha^a P \arrow[rr, "\phi(\pi_\alpha, \lambda)"] & & B(H \otimes l^2(P)), \end{tikzcd} \qquad \begin{tikzcd} & P \arrow[dl, "j_P"'] \arrow[dr, "\lambda"] & \\ \mathcal{A} \rtimes_\alpha^a P \arrow[rr, "\phi(\pi_\alpha, \lambda)"] & & B(H \otimes l^2(P)) \end{tikzcd}$$

are commutative.

Now *the reduced semicrossed product C^*-algebra* is the subalgebra in $B(H \otimes l^2(P))$ defined in the following way:

$$\mathcal{A} \rtimes_{\alpha,r}^a P := \phi_{(\pi_\alpha,\lambda)}(\mathcal{A} \rtimes_\alpha^a P).$$

We note that the homomorphisms $j_\mathcal{A} : \mathcal{A} \longrightarrow \mathcal{A} \rtimes_\alpha^a P$ and $j_P : P \longrightarrow \mathcal{A} \rtimes_\alpha^a P$ are injective and the C^*-algebra $\mathcal{A} \rtimes_\alpha^a P$ is generated by all products $j_\mathcal{A}(a)j_P(p)$, where $a \in \mathcal{A}$, $p \in P$ (see [5, Prop. 1.1]). Moreover, since the C^*-algebra $\mathcal{A}$ is unital, it is easy to see that $\mathcal{A} \rtimes_\alpha^a P$ is generated by the set $\{j_\mathcal{A}(a) | a \in \mathcal{A}\} \cup \{j_P(p) | p \in P\}$. Really, we have

$$j_\mathcal{A}(a) = (j_\mathcal{A}(I)j_P(p))^* j_\mathcal{A}(\alpha_p(a))j_P(p), \quad j_P(p) = j_\mathcal{A}(I)j_P(p),$$

where $a \in \mathcal{A}$, $p \in P$ and I is the unit of the C^*-algebra $\mathcal{A}$.

Since $\phi_{(\pi_\alpha,\lambda)}(j_\mathcal{A}(a)) = \pi_\alpha(a)$ and $\phi_{(\pi_\alpha,\lambda)}(j_P(p)) = \lambda(p)$, the C^*-algebra $\mathcal{A} \rtimes_{\alpha,r}^a P$ is generated by the set $\{\pi_\alpha(a) | a \in \mathcal{A}\} \cup \{\lambda(p) | p \in P\}$.

Let $(\mathcal{A}, P, \alpha)$ and $(\widetilde{\mathcal{A}}, P, \widetilde{\alpha})$ be two C^*-dynamical semisystems by automorphisms. An isomorphism of C^*-algebras $\sigma : \mathcal{A} \longrightarrow \widetilde{\mathcal{A}}$ is called *P-equivariant*, if for every $p \in P$ the following diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\alpha_P"] \arrow[d, "\sigma"'] & \mathcal{A} \arrow[d, "\sigma"] \\ \widetilde{\mathcal{A}} \arrow[r, "\widetilde{\alpha}_P"] & \widetilde{\mathcal{A}} \end{tikzcd}$$

is commutative, i.e., $\sigma(\alpha_p(a)) = \widetilde{\alpha}_p(\sigma(a))$ for all $a \in \mathcal{A}$. Note, since α_p and $\widetilde{\alpha}_p$ are automorphisms, we have

$$\sigma \circ \alpha_p^{-1} = \widetilde{\alpha}_p^{-1} \circ \sigma. \tag{4}$$

In this case, the semicrossed products $(\mathcal{A} \rtimes_\alpha^a P, j_\mathcal{A}, j_P)$ and $(\widetilde{\mathcal{A}} \rtimes_{\widetilde{\alpha}}^a P, \widetilde{j}_{\widetilde{\mathcal{A}}}, \widetilde{j}_P)$ are canonically isomorphic. In other words, there is a unique isomorphism of C^*-algebras $\psi : \mathcal{A} \rtimes_\alpha^a P \longrightarrow \widetilde{\mathcal{A}} \rtimes_{\widetilde{\alpha}}^a P$ such that $\psi \circ j_\mathcal{A} = \widetilde{j}_{\widetilde{\mathcal{A}}} \circ \sigma$ and $\psi \circ j_P = \widetilde{j}_P$ [20].

For the reduced semicrossed product C^*-algebras we have

Proposition 1. *Let $(\mathcal{A}, P, \alpha)$ and $(\widetilde{\mathcal{A}}, P, \widetilde{\alpha})$ be C^*-dynamical semisystems by automorphisms. Let $\sigma : \mathcal{A} \longrightarrow \widetilde{\mathcal{A}}$ be a P-equivariant isomorphism of C^*-algebras. Then, the following equality holds:*

$$\mathcal{A} \rtimes_{\alpha,r}^a P = \widetilde{\mathcal{A}} \rtimes_{\widetilde{\alpha},r}^a P.$$

Proof. Let $\pi : \mathcal{A} \longrightarrow B(H)$ be a faithful representation of C^*-algebra $\mathcal{A}$ on a Hilbert space H. Let us define the faithful representation of C^*-algebra $\widetilde{\mathcal{A}}$ on the Hilbert space H by formula $\widetilde{\pi} := \pi \circ \sigma^{-1}$. Then, both C^*-algebras $\mathcal{A} \rtimes^{a}_{\alpha,r} P$ and $\widetilde{\mathcal{A}} \rtimes^{a}_{\widetilde{\alpha},r} P$ are subalgebras in $B(H \otimes l^2(P))$.

We claim that these subalgebras coincide. Indeed, using (3) and (4) we get

$$\widetilde{\pi}_{\widetilde{\alpha}}(\sigma(a))(\xi \otimes e_h) = \widetilde{\pi}(\widetilde{\alpha}_h^{-1}(\sigma(a)))\xi \otimes e_h = \widetilde{\pi}(\sigma(\alpha_h^{-1}(a)))\xi \otimes e_h$$
$$= \pi(\alpha_h^{-1}(a))\xi \otimes e_h = \pi_{\alpha}(a)(\xi \otimes e_h)$$

for every $a \in \mathcal{A}, \xi \in H, h \in P$. Since σ is an isomorphism, the following sets coincide

$$\{\pi_{\alpha}(a)|a \in \mathcal{A}\} = \{\widetilde{\pi}_{\widetilde{\alpha}}(a')|a' \in \widetilde{\mathcal{A}}\}.$$

The equality of sets $\{\lambda(p)|p \in P\}$ and $\{\widetilde{\lambda}(p)|p \in P\}$ is obvious.

Because the generating sets of the C^*-algebras $\mathcal{A} \rtimes^{a}_{\alpha,r} P$ and $\widetilde{\mathcal{A}} \rtimes^{a}_{\widetilde{\alpha},r} P$ are the same, the algebras coincide, as claimed. □

Remark 1. Let tr denote both the trivial homomorphism $\mathrm{tr} : P \longrightarrow \mathrm{Aut}(\mathcal{A})$ and $\mathrm{tr} : P \longrightarrow \mathrm{Aut}(\widetilde{\mathcal{A}})$ taking each element of P to the identity automorphism of the C^*-algebras $\mathcal{A}$ and $\widetilde{\mathcal{A}}$. Then, every isomorphism $\sigma : \mathcal{A} \longrightarrow \widetilde{\mathcal{A}}$ is a P-equivariant isomorphism for the C^*-dynamical semisystems $(\mathcal{A}, P, \mathrm{tr})$ and $(\widetilde{\mathcal{A}}, P, \mathrm{tr})$, and we have the equality

$$\mathcal{A} \rtimes^{a}_{\mathrm{tr},r} P = \widetilde{\mathcal{A}} \rtimes^{a}_{\mathrm{tr},r} P.$$

3. THE SEMIGROUP C^*-ALGEBRa $C_r^*(S \rtimes_\beta P)$

Let S and P be discrete left cancelative semigroups. Let $\beta : P \longrightarrow \mathrm{Aut}(S)$ be a semigroup homomorphism. Let us define the semidirect product of semigroups S and P. To emphasize that the semigroup P acts on S by automorphisms, we use the notation $S \rtimes^a_\beta P$. So the semidirect product $S \rtimes^a_\beta P$ is the semigroup with the underlying set $S \times P$ and the semigroup operation given by

$$(a, p)(b, q) := (a\beta_p(b), pq),$$

where $a, b \in S$, $p, q \in P$. It is easy to see that the semigroup $S \rtimes^a_\beta P$ has the left cancelation property. Here the object of our study is the reduced semigroup C^*-algebra $C_r^*(S \rtimes^a_\beta P)$. We fix arbitrary elements $s, t \in S$ and $x, y \in P$. Let us introduce the notation

$$V_{a,s,x} := T^*_{(s,x)} T_{(s\beta_x(a),x)} \quad \text{and} \quad W_{p,t,y} := T^*_{(t,y)} T_{(t,yp)},$$

where $a \in S, p \in P$. We show that the actions of the operators $V_{a,s,x}$ and $W_{p,t,y}$ on the space $l^2(S \rtimes^a_\beta P)$ do not depend on the choice of the elements s, t, x, y. To do this, we find out how these operators act on the basis vectors. We have

$$V_{a,s,x} e_{(b,q)} = T^*_{(s,x)} T_{(s\beta_x(a),x)} e_{(b,q)} = T^*_{(s,x)} e_{(s\beta_x(a)\beta_x(b),xq)} = T^*_{(s,x)} T_{(s,x)} e_{(ab,q)} = e_{(ab,q)}, \tag{5}$$

$$W_{p,t,y} e_{(b,q)} = T^*_{(t,y)} T_{(t,yp)} e_{(b,q)} = T^*_{(t,y)} e_{(t\beta_{yp}(b),ypq)} = T^*_{(t,y)} T_{(t,y)} e_{(\beta_p(b),pq)} = e_{(\beta_p(b),pq)}, \tag{6}$$

where $a, b \in S$, $p, q \in P$. Thus, the actions of the operators $V_{a,s,x}$ and $W_{p,t,y}$ on the basis vectors do not depend on the elements s, x and t, y. So the operators $V_{a,s,x}$ and $W_{p,t,y}$ are denoted by V_a and W_p, respectively.

Lemma 1. *The following properties are fulfilled:*

1) *The operators V_a and W_p are isometries for every $a \in S$ and $p \in P$;*

2) *The C^*-algebra $C_r^*(S \rtimes^a_\beta P)$ is generated by the set of isometries $\{V_a|a \in S\} \cup \{W_p|p \in P\}$.*

Proof. 1) Firstly, we calculate the values of the operator V_a^* at the basis vectors. We note that

$$V_a^* e_{(b,q)} = T^*_{(s\beta_x(a),x)} T_{(s,x)} e_{(b,q)} = T^*_{(s\beta_x(a),x)} e_{(s\beta_x(b),xq)} \neq 0,$$

if $(s\beta_x(b), xq) = (s\beta_x(a), x)(u, v)$ for some element $(u, v) \in S \rtimes^a_\beta P$. This implies the equality $(s\beta_x(b), xq) = (s\beta_x(au), xv)$. Since S and P are left cancelative semigroups, the last equality is possible if and only if $b = au$ and $v = q$. Thus, we have

$$V_a^* e_{(b,q)} = \begin{cases} e_{(u,q)}, & \text{if } b = au; \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

where $a, b, u \in S$, $q \in P$. Next, using (5) and (7), we get $V_a^* V_a = I$.

Secondly, we calculate the values of the operator W_p^* at the basis vectors. We have

$$W_p^* e_{(b,q)} = T^*_{(t,yp)} T_{(t,y)} e_{(b,q)} = T^*_{(t,yp)} e_{(t\beta_y(b),yq)} \neq 0,$$

if $(t\beta_y(b), yq) = (t, yp)(u, v)$ for some $(u, v) \in S \rtimes_\beta P$. This implies the equality $(t\beta_y(b), yq) = (t\beta_{yp}(u), ypv)$. Since S and P are left cancelative semigroups, the last equality is possible if and only if $b = \beta_p(u)$ and $q = pv$. Thus, we have

$$W_p^* e_{(b,q)} = \begin{cases} e_{(\beta_p^{-1}(b),v)}, & \text{if } q = pv; \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

where $b \in S$, $p, q, v \in P$. Next, using (6) and (8), we get $W_p^* W_p = I$.

2) Let us show that for any $a \in S$, $p \in P$ the equality $T_{(a,p)} = V_a W_p$ holds. Really, using (5) and (6), we get

$$V_a W_p e_{(b,q)} = V_a e_{(\beta_p(b),pq)} = e_{(a\beta_p(b),pq)} = T_{(a,p)} e_{(b,q)},$$

where $a, b \in S$, $p, q \in P$. $\square$

Further, we consider the C^*-algebra $C_r^*(S)$. For constructing the C^*-dynamical semisystem $(C_r^*(S), P, \alpha)$ by automorphisms we use the following

Lemma 2 [15]. *Let $\gamma : S \longrightarrow S$ be an automorphism of the semigroup S. Then, there exists a unique automorphism $\overline{\gamma} : C_r^*(S) \longrightarrow C_r^*(S)$ such that $\overline{\gamma}(T_a) = T_{\gamma(a)}$ whenever $a \in S$.*

Thus, if $\beta : P \longrightarrow \mathrm{Aut}(S)$ is a semigroup homomorphism, then we have the semigroup homomorphism $\alpha : P \longrightarrow \mathrm{Aut}(C_r^*(S))$ such that $\alpha_p(T_a) = T_{\beta_p(a)}$ for all $p \in P$, $a \in S$. So we have the C^*-dynamical semisystem $(C_r^*(S), P, \alpha)$ by automorphisms. We note, since α_p and β_p are automorphisms, it is easy to see that $\alpha_p^{-1}(T_a) = T_{\beta_p^{-1}(a)}$ for all $p \in P$, $a \in S$.

Next, let us construct the reduced semicrossed product $C_r^*(S) \rtimes^a_{\alpha,r} P$ by automorphisms. Firstly, using the inclusion $C_r^*(S) \subset B(l^2(S))$, we define the representation $\pi : C_r^*(S) \longrightarrow B(l^2(S) \otimes l^2(P))$ on the generators of the C^*-algebra $C_r^*(S)$ as follows

$$\pi(T_a)(e_b \otimes e_q) = \alpha_q^{-1}(T_a) e_b \otimes e_q = e_{\beta_q^{-1}(a)b} \otimes e_q, \tag{9}$$

where $a, b \in S$, $q \in P$. Secondly, we define the regular representation $\lambda : P \longrightarrow B(l^2(S) \otimes l^2(P))$ by

$$\lambda(p)(e_b \otimes e_q) = e_b \otimes e_{pq}, \tag{10}$$

where $b \in S$, $p, q \in P$. Then, the pair (π, λ) is a covariant representation of the C^*-dynamical semisystem $(C_r^*(S), P, \alpha)$ by automorphisms. So we have the reduced semicrossed product

$$C_r^*(S) \rtimes^a_{\alpha,r} P = \phi_{(\pi,\lambda)}(C_r^*(S) \rtimes^a_\alpha P).$$

The C^*-algebra $C_r^*(S) \rtimes^a_{\alpha,r} P$ is generated by the set $\{\pi(A) | A \in C_r^*(S)\} \cup \{\lambda(p) | p \in P\}$. Therefore, because the C^*-algebra $C_r^*(S)$ is generated by the set of operators $\{T_a | a \in S\}$, one can see that the C^*-algebra $C_r^*(S) \rtimes^a_{\alpha,r} P$ is generated by the set $\{\pi(T_a) | a \in S\} \cup \{\lambda(p) | p \in P\}$.

Theorem 1. *Let S and P be discrete left cancelative semigroups. Let $\beta : P \longrightarrow Aut(S)$ and $\alpha : P \longrightarrow Aut(C_r^*(S))$ be semigroup homomorphisms such that $\alpha_p(T_a) = T_{\beta_p(a)}$ for all $p \in P$, $a \in S$. Then, there exists an isomorphism of C^*-algebras*

$$C_r^*(S \rtimes_\beta^a P) \cong C_r^*(S) \rtimes_{\alpha,r}^a P.$$

Proof. Let us consider the operator $U : l^2(S) \otimes l^2(P) \longrightarrow l^2(S \rtimes_\beta^a P)$ defined by the formula

$$U(e_a \otimes e_p) = e_{(\beta_p(a),p)}, \tag{11}$$

where $a \in S$, $p \in P$. Since β_p is the automorphism, the operator U is unitary.

Furthermore, we claim that the following diagrams are commutative

$$\begin{tikzcd} l^2(S) \otimes l^2(P) \arrow[r, "\lambda_{(p)}"] \arrow[d, "U"'] & l^2(S) \otimes l^2(P) \arrow[d, "U"] \\ l^2(S \rtimes_\beta^a P) \arrow[r, "W_P"] & l^2(S \rtimes_\beta^a P) \end{tikzcd}$$

for every $p \in P$, and

$$\begin{tikzcd} l^2(S) \otimes l^2(P) \arrow[r, "\pi(T_a)"] \arrow[d, "U"'] & l^2(S) \otimes l^2(P) \arrow[d, "U"] \\ l^2(S \rtimes_\beta^a P) \arrow[r, "V_a"] & l^2(S \rtimes_\beta^a P) \end{tikzcd}$$

for every $a \in S$.

Indeed, using (6) and (11), we get

$$W_p U(e_b \otimes e_q) = W_p e_{(\beta_q(b),q)} = e_{(\beta_p(\beta_q(b)),pq)} = e_{(\beta_{pq}(b),pq)},$$

where $b \in S$, $p, q \in P$. On the other hand, by (10) and (11), we have

$$U\lambda(p)(e_b \otimes e_q) = U(e_b \otimes e_{pq}) = e_{(\beta_{pq}(b),pq)}.$$

Thus, the commutativity of the first diagram is shown.

Consider the second diagram. On the one hand, using (5), we have

$$V_a U(e_b \otimes e_q) = V_a e_{(\beta_q(b),q)} = e_{(a\beta_q(b),q)},$$

where $a, b \in S$, $q \in P$. On the other hand, using (9), we get

$$U\pi(T_a)(e_b \otimes e_q) = U(e_{\beta_q^{-1}(a)b} \otimes e_q) = e_{(\beta_q(\beta_q^{-1}(a)b),q)} = e_{(a\beta_q(b),q)}.$$

The commutativity of the second diagram is proved, as claimed.

Therefore, the equalities

$$\lambda(p) = U^* W_p U, \quad \pi(T_a) = U^* V_a U \tag{12}$$

are true for all $p \in P$ and $a \in S$, respectively.

Further, we define the isometric $*$-homomorphism

$$\phi : C_r^*(S \rtimes_\beta^a P) \longrightarrow B(l^2(S) \otimes l^2(P)) : A \longmapsto U^* A U,$$

where $A \in C_r^*(S \rtimes_\beta^a P)$. By (12), we have

$$\phi(W_p) = \lambda(p), \quad \phi(V_a) = \pi(T_a)$$

whenever $p \in P$ and $a \in S$.

The image of ϕ is dense in the C^*-algebra $C_r^*(S) \rtimes_{\alpha,r}^a P$. It follows from the fact that the C^*-algebra $C_r^*(S) \rtimes_{\alpha,r}^a P$ is generated by the set $\{\pi(T_a) | a \in S\} \cup \{\lambda(p) | p \in P\}$. Thus, the homomorphism ϕ realizes the required isomorphism between the C^*-algebras $C_r^*(S \rtimes_\beta^a P)$ and $C_r^*(S) \rtimes_{\alpha,r}^a P$. $\square$

4. EXAMPLES

In this section, we give several examples of applications of Theorem 1.

Example 1. Let tr denote both the trivial homomorphism $\mathrm{tr} : P \longrightarrow \mathrm{Aut}(C_r^*(S))$ taking each element of P to the identity automorphism of the C^*-algebra $C_r^*(S)$ and the trivial action $\mathrm{tr} : P \longrightarrow \mathrm{Aut}(S)$ of the semigroup P on the semigroup S. Consider the Cartesian product $S \times P$. Here we treat $S \times P$ as the semigroup with the coordinatewise binary operation. Obviously, we have the equality $S \times P = S \rtimes^a_{\mathrm{tr}} P$. Then, using Theorem 1, we obtain

$$C_r^*(S \times P) \cong C_r^*(S) \rtimes^a_{\mathrm{tr},r} P. \tag{13}$$

We note that the C^*-algebra $C_r^*(S) \rtimes^a_{\mathrm{tr},r} P \subset B(l^2(S) \otimes l^2(P))$ is generated by the set of operators $\{\pi_{\mathrm{tr}}(T_a) | a \in S\} \cup \{\lambda(p) | p \in P\}$, which act on basis vectors as follows

$$\pi_{\mathrm{tr}}(T_a)(e_b \otimes e_q) = T_a e_b \otimes e_q = e_{ab} \otimes e_q,$$
$$\lambda(p)(e_b \otimes e_q) = e_b \otimes e_{pq},$$

where $a, b \in S, p, q \in P$. Thus, it is easy to see, that there exists an isomorphism of C^*-algebras

$$C_r^*(S) \rtimes^a_{\mathrm{tr},r} P \cong C_r^*(S) \otimes_{min} C_r^*(P).$$

Example 2. Let us consider the reduced semigroup C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$, which is studied in [15–18]. In this and the next two examples we will get different representations of this C^*-algebra as semicrossed product C^*-algebras.

Here $\mathbb{Z}$ is the additive group of all integers and $\mathbb{Z}^\times$ is the multiplicative semigroup $\mathbb{Z} \setminus \{0\}$. Let $\varphi : \mathbb{Z}^\times \longrightarrow \mathrm{Aut}(\mathbb{Z})$ be the semigroup homomorphism from $\mathbb{Z}^\times$ into the semigroup of automorphisms of the group $\mathbb{Z}$ given by

$$\varphi_m(n) := \begin{cases} n, & \text{if } m > 0; \\ -n, & \text{if } m < 0, \end{cases}$$

where $m \in \mathbb{Z}^\times$, $n \in \mathbb{Z}$. So $\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times$ is the semidirect product of $\mathbb{Z}$ and $\mathbb{Z}^\times$ with respect to φ. It is a cancellative semigroup with respect to the multiplication defined by

$$(m, n)(k, l) = (m + \varphi_n(k), nl),$$

where $m, k \in Z, n, l \in Z^\times$.

Theorem 1 implies the following representation for the reduced semigroup C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$

$$C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times) \cong C_r^*(\mathbb{Z}) \rtimes^a_{\phi,r} \mathbb{Z}^\times, \tag{14}$$

where $\phi_m(U_n) = U_{\varphi_m(n)}$, $n \in \mathbb{Z}$, $m \in \mathbb{Z}^\times$. Here the operator $U_n \in B(l^2(\mathbb{Z}))$ is unitary and $U_n^* = U_{-n}$, $n \in \mathbb{Z}$. Therefore, we have $\phi_m = id$ for $m > 0$ and $\phi_m(U_n) = U_n^*$ for $m < 0$.

Since the group $\mathbb{Z}$ is amenable, the C^*-algebra $C_r^*(\mathbb{Z})$ is the universal C^*-algebra generated by the unitary element $u := U_1$, that is

$$C_r^*(\mathbb{Z}) = C^*(\mathbb{Z}) = C^*\{u | u^*u = uu^* = I\}. \tag{15}$$

It is known that the C^*-algebra $C^*(\mathbb{Z})$ is isomorphic to the C^*-algebra $C(S^1)$ of all continuous complex-valued functions on the unit circle in the complex plane (see, for example, [14, Appendix]). Let us consider the C^*-dynamical semisystem $(C(S^1), \mathbb{Z}^\times, \widetilde{\phi})$ by automorphisms, where $\widetilde{\phi}_m = id$ for $m > 0$ and $\widetilde{\phi}_m(f) = f^*$ for $m < 0$ whenever $m \in \mathbb{Z}^\times$, $f \in C(S^1)$. It is not difficult to see that the isomorphism

$$\sigma : C^*(\mathbb{Z}) \longrightarrow C(S^1) : u \mapsto \{z : e^{i\theta} \mapsto e^{i\theta}, 0 \leq \theta < 2\pi\} \tag{16}$$

is $\mathbb{Z}^\times$-equivariant for C^*-dynamical semisystems $(C(S^1), \mathbb{Z}^\times, \widetilde{\phi})$ and $(C^*(\mathbb{Z}), \mathbb{Z}^\times, \phi)$. Therefore, using Proposition 1 and the isomorphism (14), we have the following representation for the reduced semigroup C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$:

$$C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times) \cong C(S^1) \rtimes^a_{\widetilde{\phi},r} \mathbb{Z}^\times.$$

Example 3. Now, we will show that the representation

$$C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times) \cong (C(S^1) \rtimes_{\widetilde{\alpha}} \mathbb{Z}_2) \rtimes^a_{\mathrm{tr},r} \mathbb{N} \tag{17}$$

holds, where $\mathbb{N}$ is the multiplicative semigroup of natural numbers, $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} = \{0,1\}$ is the cyclic group of order two and $\widetilde{\alpha}_0 = id$, $\widetilde{\alpha}_1(f) = f^*$, $f \in C(S^1)$.

Indeed, it was shown in the paper [18] that the semigroup $\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times$ is isomorphic to the semigroup $D_\infty \times \mathbb{N}$, where D_∞ is the infinite dihedral group. Therefore, using (13), we get the isomorphism of C^*-algebras

$$C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times) \cong C_r^*(D_\infty) \rtimes^a_{\mathrm{tr},r} \mathbb{N}. \tag{18}$$

Further, by (2), we have the isomorphism of C^*-algebras

$$C_r^*(D_\infty) \cong C^*(\mathbb{Z}) \rtimes_\alpha \mathbb{Z}_2,$$

where $\alpha_0 = id$, $\alpha_1(u) = u^*$.

It is easy to check that the isomorphism σ, given by the formula (16), is $\mathbb{Z}_2$-equivariant for C^*-dynamical systems $(C^*(\mathbb{Z}), \mathbb{Z}_2, \alpha)$ and $(C(S^1), \mathbb{Z}_2, \widetilde{\alpha})$. Hence, using Proposition 1, we have the isomorphism of C^*-algebras

$$C_r^*(D_\infty) \cong C(S^1) \rtimes_{\widetilde{\alpha}} \mathbb{Z}_2. \tag{19}$$

Finally, the representation (17) follows from the formulas (18), (19) and Remark 1.

Example 4. In this example, we will show that in the formula (17) we can change the places of the semigroup $\mathbb{N}$ and the group $\mathbb{Z}_2$. Namely, we will show that the representation

$$C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times) \cong (C(S^1) \rtimes^a_{\mathrm{tr},r} \mathbb{N}) \rtimes_{\widetilde{\gamma}} \mathbb{Z}_2 \tag{20}$$

holds, where $\widetilde{\gamma}_0 = id$ and the automorphism $\widetilde{\gamma}_1$ is defined by the action on the generating elements of the C^*-algebra $C(S^1) \rtimes^a_{\mathrm{tr},r} \mathbb{N}$ as follows: $\widetilde{\gamma}_1(\pi_{\mathrm{tr}}(f)) = \pi_{\mathrm{tr}}(f)^*$, $\widetilde{\gamma}_1(\lambda(n)) = \lambda(n)$, where $f \in C(S^1)$, $n \in \mathbb{N}$.

Let $\mathbb{Z} \times \mathbb{N}$ be the Cartesian product of the additive group of all integers and the multiplicative semigroup of the natural numbers. It is a semigroup under the multiplication $(m,n)(k,l) = (m+k, nl)$, where $m, k \in \mathbb{Z}$, $n, l \in \mathbb{N}$. In [15, 17], we consider the C^*-dynamical system $(C_r^*(\mathbb{Z} \times \mathbb{N}), \mathbb{Z}_2, \gamma)$, where $\gamma_0 = id$ and γ_1 is defined by the action on the generating elements of the C^*-algebra $C_r^*(\mathbb{Z} \times \mathbb{N})$ in the following way: $\gamma_1(T_{(m,n)}) = T_{(-m,n)}$ for all $m \in \mathbb{Z}$, $n \in \mathbb{N}$. In particular, it is proved there that there exists an isomorphism

$$C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times) \cong C_r^*(\mathbb{Z} \times \mathbb{N}) \rtimes_\gamma \mathbb{Z}_2. \tag{21}$$

Further, using (13), (15), (16) and Remark 1, we get the isomorphism of C^*-algebras

$$\tau : C_r^*(\mathbb{Z} \times \mathbb{N}) \longrightarrow C(S^1) \rtimes^a_{\mathrm{tr},r} \mathbb{N},$$

where $\tau(T_{(m,n)}) = \pi_{\mathrm{tr}}(z)^m \lambda(n)$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$ and $\{z : e^{i\theta} \mapsto e^{i\theta}, 0 \le \theta < 2\pi\} \in C(S^1)$.

It is not difficult to verify that the isomorphism τ is $\mathbb{Z}_2$-equivariant for the C^*-dynamical systems $(C_r^*(\mathbb{Z} \times \mathbb{N}), \mathbb{Z}_2, \gamma)$ and $(C(S^1) \rtimes^a_{\mathrm{tr},r} \mathbb{N}, \mathbb{Z}_2, \widetilde{\gamma})$. Hence, we have the isomorphism of C^*-algebras

$$C_r^*(\mathbb{Z} \times \mathbb{N}) \rtimes_\gamma \mathbb{Z}_2 \cong (C(S^1) \rtimes^a_{\mathrm{tr},r} \mathbb{N}) \rtimes_{\widetilde{\gamma}} \mathbb{Z}_2. \tag{22}$$

Finally, the representation (20) follows from the formulas (21) and (22).

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CONFLICT OF INTEREST

The author of this work declares that she has no conflicts of interest.

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