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Algebraic Model of Non-Abelian Superselection Rules Considering Conjugate Endomorphism

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Abstract—In this paper, we consider an extension of the previously proposed algebraic model and study the constraints of non-Abelian superselection rules on the transfer quantum information, taking into account conjugate endomorphism. The procedure of averaging (over the group $G = SU(3)$) projectors to the basic states of coherent orthogonal subspaces into which the space of two three-level systems decomposes is considered. Main attention is paid to the superselection structure of the algebra of observables ${}^0O_G$ defined by the Cuntz algebra ${}^0O_{d=3}$ (field algebra) containing ${}^0O_G$ as a pointwise fixed subalgebra with respect to the action of the gauge group G. As an application of the model, we consider the encoding of information using a three-level system and show that information can be transmitted only by those states whose projectors belong to the algebra of observables. These projectors commute with the elements of the representation of the group G, and therefore, allow the recipient to restore the obtained information.

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1. INTRODUCTION

Over the past 25–30 years, quantum theory has developed so rapidly and successfully that whole concepts have emerged that can be called quantum technologies and quantum engineering. For example, quantum sensing is developing rapidly [1], the properties of charge and spin qubits based on semiconductor quantum dots are being intensively studied [2], multi-qubit (50–100) processors for quantum computers have been created and in this regard, there has been a rise of interest in superconducting qubits [3, 4], research on the transmission of quantum information and quantum cryptography, etc., is being conducted on a broad front. Such success also initiated the formulation of a number of original experiments to answer a number of fundamental questions together with rapid progress in experimental technology. In particular, the experimental possibility of obtaining and controlling individual quantum mechanical states now allows us to rethink some well-known positions of quantum mechanics. Therefore, since the end of the 90s of the last century many purely academic issues began to be discussed at a serious level and new questions began to be raised about the nature of decoherence, quantum entanglement and quantum measurements of — phenomena that form the basis of quantum technologies. However, despite some successes achieved in this area, there remain a number of problems that require, along with a detailed analysis of experimental data, also the development of a theoretical framework based on general physical principles, taking into account the fundamental laws of nature.

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One of such fundamental laws of the quantum world is the dynamical superselection rules generated by internal symmetries and associated with absolutely conserved Abelian or non-Abelian charges. For example, in some aspects their role in the theory of quantum entanglement is studied in [5–7], and in the theory of transmission and security of quantum information — in [8–12]. Note that in [11, 12] much attention is paid to the non-Abelian dynamical superselection rules.

The internal symmetries generating the dynamical superselection rules are described using compact topological groups G, which in elementary particle physics correspond to groups of global gauge transformations. In [13], we proposed an algebraic model to study the role of non-Abelian superselection rules in the theory of quantum information transmission, which allowed us to show that information can be encoded only with the help of those states to which projectors commute with the algebra of observables. The model was based on an abstract symmetric tensor C^*-category (isomorphic to category of finite-dimensional Hilbert spaces), which, according to the Doplicher–Roberts duality, is a dual object to the compact group G. Objects of this category are endomorphisms of C^*-algebra of observables $\mathcal{A}$, and morphisms (arrows) are the intertwining operators between them. From a physical point of view, they can be called non-Abelian charges (see paragraphs 2.2 and 3.1). However, non-Abelian conjugate charges were not considered in this model. Although the need to take into account such conjugate charges within the framework of the quantum theory of information transfer has already been expressed in [11], however, their detailed study (for the non-Abelian case) is not yet available. It would also be interesting to identify their role in other mentioned fundamental phenomena. For example, it is relevant to study the appearance of a mutual geometric (topological) phase when considering the evolution of an entangled state [14], and some aspects of the Abelian charge superselection rules when studying geometric phases are considered, for example, in [15]. Also in [16], the appearance of sectors having a topological nature was studied in the study of the topological phase in a nucleon system.

In [17] we constructed a model of a symmetric tensor C^*-category with conjugation at the dimension of the object $d = 3$ and proved that the constructed conjugate object satisfies the conjugation equations. The purpose of this work is to generalize the model proposed in [13] using this conjugate object. Therefore, the model allows us to simultaneously investigate the superselection rules generated by conjugate non-Abelian charges.

In accordance with this, the work is structured as follows. Section 2 is preliminary in nature, where we provide the information necessary for further presentation. The third section is original, and here we consider the scheme of averaging over a group of $SU(3)$ projectors into the basic states of the spaces $\bar{\mathcal{H}}_3$ and $\mathcal{H}_6$, into which the state space of two three-level systems is decomposed. Here we also study the algebra of observables of this system and the statistics of the corresponding sectors. As an illustration, we apply the results to the process of transmitting quantum information using a three-level system. In conclusion, brief conclusions are made. In order to ensure the greatest independence from the cited sources on the theory of C^*-categories and related issues, we provide information about tensor symmetric C^*-categories in the appendix.

2. ALGEBRAIC MODEL

2.1. Background

One of the mathematically rigorous ways of describing quantum physical systems is based on the analysis of the quasi-local C^*-algebra of observables $\mathcal{A}$ [18–20]. As it was shown in [21, 22], the algebra of such observables can be embedded in a consistent way into another, extended algebra obtained on the basis of the crossed product technique $\mathcal{A} \times \mathcal{T}$, where $\mathcal{T}$ is an abstract symmetric tensor C^*-category closed with respect to direct sums and subobjects. In the literature, such a category is called the Doplicher–Roberts category and its exhaustive description is made in [23] (see also the Appendix). The specified crossed product corresponds to the field C^*-algebra $\mathfrak{F} = \mathcal{A} \times \mathcal{T}$, where as the automorphism group $aut(\mathfrak{F})$ is a compact group G, and $\mathfrak{F}$ contains the C^*-algebra $\mathcal{A}$ as its pointwise fixed subalgebra with respect to the action of this group [22]. The physical meaning of the group G corresponds to the group of global gauge transformations.

In [23] it is shown that the category $\mathcal{T}$ can be embedded $\mathcal{T} \subset \mathbf{end}(\mathcal{A})$ as a complete subcategory in the category of endomorphisms of C^*-algebra $\mathcal{A}$. If we restrict ourselves to the subcategory $\mathcal{T}_\rho \subset \mathcal{T}$ generated by tensor powers of one endomorphism ρ, then, as shown in [21], $\mathcal{A} \times \mathcal{T}_\rho$ is a field algebra whose subalgebra is the Cuntz C^*-algebra [24]. The automorphism group of the Cuntz algebra is the compact Lie group. The Cuntz algebra O_d is generated by isometric operators ψ_i $(i = 1, 2, ..., d)$ satisfying the relations

$$\psi_i^* \psi_j = \delta_{ij} I \tag{1}$$

and

$$\sum_i^d \psi_i \psi_i^* = I, \tag{2}$$

where I is the unit of the Cuntz algebra (note that ψ_i^* are not isometries). The linear span of these isometries forms the so-called canonical Hilbert space $\mathcal{H} = \mathrm{Lin}\{\psi_i\}_{i=1}^d$[25]. In this case, the isometries are given an orthonormal basis $\{\psi_i\}_{i=1,2,...,d}$, and the scalar product in such a complex Hilbert space is given by the relation

$$\psi^* \psi' = \langle \psi, \psi' \rangle I, \quad \psi, \psi' \in \mathcal{H}. \tag{3}$$

Therefore, Hilbert subspaces can be considered in the Cuntz algebra, and their tensor product $\underbrace{\mathcal{H} \otimes ... \otimes \mathcal{H}}_{r} = \mathcal{H}^r$ can be identified with the elementwise product of these subspaces in the Cuntz algebra. Since the operators $\mathcal{H}^s \to \mathcal{H}^r \equiv (\mathcal{H}^s, \mathcal{H}^r)$ between tensor degrees are given using linear maps of the form

$$t = \psi_{i_1} ... \psi_{i_r} \psi_{j_s}^* ... \psi_{j_1}^* \in (\mathcal{H}^s, \mathcal{H}^r), \tag{4}$$

which form a complex Banach space, then it is possible to define a certain $*$-algebra ${}^0\mathcal{O}_d$ as a direct sum of ${}^0\mathcal{O}_d = \oplus_k^0 \mathcal{O}_d^k$ of inductive limits

$$(\mathcal{H}^r, \mathcal{H}^{r+k}) \longrightarrow^{\otimes 1} (\mathcal{H}^{r+1}, \mathcal{H}^{r+1+k}) \longrightarrow^{\otimes 1} ... \longrightarrow^{\otimes 1} (\mathcal{H}^{r+n}, \mathcal{H}^{r+n+k}) \longrightarrow^{\otimes 1} .... \tag{5}$$

Here ${}^0\mathcal{O}_d^k$ is the Banach space, which is the inductive limit (5) of Banach spaces with fixed values $k \in \mathbb{Z}$. From (4) and (5) we see that the mappings $t \to t \otimes 1$ are injective. Completion of this algebra by a unique C^*-norm leads to the Cuntz algebra $\mathcal{O}_d$ [25].

In this case, G-invariant operators $(\mathcal{H}^r, \mathcal{H}^s)_G$ generate an algebra of observables, i.e., a pointwise fixed subalgebra $\mathcal{O}_G$ of the Cuntz algebra. Also note that, in general, $\mathcal{O}_G \subseteq \mathcal{A}$.

As it was shown in [21, 25], the generators of the algebra $\mathcal{O}_G$ for an arbitrary d are operators of the form

$$\vartheta(p_n) = \sum_{i_1 i_2 ... i_n} \psi_{i_1} ... \psi_{i_n} \psi_{i_{p(n)}}^* ... \psi_{i_{p(1)}}^*, \tag{6}$$

where $p_n \in \mathbf{P}_n$ and

$$S = \frac{1}{\sqrt{d!}} \sum_{p \in \mathbf{P}_d} \mathrm{sign}(p) \psi_{p(1)} ... \psi_{p(d)}. \tag{7}$$

Here $\mathbf{P}_n$ is a symmetric group, $\mathbf{P}_d \subset \mathbf{P}_n$.

2.2. The Structure of Superselection Sectors

In quantum systems, in the presence of an absolutely conserved quantity, the superselection rules apply, prohibiting any transitions between states with different values of the eigenvalues of the superselection operator $\mathfrak{S}$. Therefore, in these systems, the state space $\mathcal{H}$ is represented as a direct sum of $\mathcal{H} = \oplus \mathcal{H}_i$ orthogonal subspaces $\mathcal{H}_i$, called coherent superselection sectors [20]. At the same time, as noted, transitions between different orthogonal subspaces under the action of observable operators are forbidden, and superpositions of vectors from various such subspaces form mixed states. The superselection operator can be represented as

$$\mathfrak{S} = \Sigma \mu_i \Pi_i, \tag{8}$$

where Π_i are projectors to coherent subspaces $\mathcal{H}_i$, and μ_i are some real numbers [20]. The operator (8) belongs to the center of the algebra of observables $\mathcal{Z}(\mathcal{A})$. If the algebra of observables of physical system is defined using an abstract C^*-algebra $\mathcal{A}$, then the representations of this algebra $\pi(\mathcal{A})$ in subspaces $\mathcal{H}_i$ are factorial of type I and pairwise disjunct [26].

However, at the same time, the connection of the superselection structure (the set of superselection sectors) of the algebra of observables $\mathcal{A}$ with the field algebra $\mathcal{F}$ remained unclear. The decisive role in establishing this connection was played by the works [27, 28], where the theory of superselection sectors was formulated, based on criteria for distinguishing a class of physically acceptable representations of the quasilocal algebra of observables $\mathcal{A}$ (the so-called Doplicher — Haag — Roberts selection criterion, or, in short, the DHR criterion)[1)]. Relying on powerful category-theoretic methods, this formulation allowed the authors to further prove the existence of a compact group of internal symmetries G (gauge group). More specifically, in the algebra $\mathfrak{F}$ there are G-invariant Hilbert spaces $\mathcal{H} \subset \mathfrak{F}$ for which $g(\mathcal{H}) = \mathcal{H}$, $g \in G$. Such Hilbert spaces, where unitary irreducible representations of the group G are realized, form the category of representations of $\mathbf{Rep(G)}$ and correspond to superselection sectors. With each such Hilbert space an internal endomorphism $\rho_{\mathcal{H}}$ of the algebra $\mathfrak{F}$

$$\rho_{\mathcal{H}}(F) = \sum_{i=1}^{d=\dim(\mathcal{H})} \psi_i F \psi_i^*, \quad F \in \mathfrak{F}.$$

is associated.

The narrowing of this endomorphism into the algebra $\mathcal{A}$ is called canonical, which, if necessary, in the future we will simply call the endomorphism of the algebra $\mathcal{A}$ and we will denote by the letter ρ, and $\rho(a) = \sum_i \psi_i a \psi_i^*$, $a \in \mathcal{A}$. These endomorphisms form a symmetric tensor C^*-category and in the case of tensor powers of one endomorphim we have $\mathcal{T}_\rho \subset \mathbf{end}(\mathcal{A})$. Each such endomorphism can be associated with the Hilbert space $\rho \to \mathcal{H}_\rho = \{\psi \in \mathfrak{F} | \psi a = \rho(a)\psi, \quad a \in \mathcal{A}\}$ in algebra $\mathcal{A}$. Since $\mathcal{H}_\rho$ has unitary representations of u_g of the group G, the relation $u_g(g)\psi = \psi'$ is valid, where $\psi, \psi' \in \mathcal{H}_\rho$.

In [17] we showed that in $\mathcal{T}_\rho$ for $d = 3$ and a given ρ there exists a conjugate object $\bar{\rho}(a) = \sum_{i=1}^{3} \hat{\psi}_i a \hat{\psi}_i^*$ and morphisms $r = \sum_{i=1}^{3} \hat{\psi}_i \psi_i \in (\iota, \bar{\rho}\rho)$ and $\bar{r} = \sum_{i=1}^{3} \psi_i \hat{\psi}_i \in (\iota, \rho\bar{\rho})$ satisfying conjugation equations, so that the category $\mathcal{T}_\rho$ is a category with conjugation. Here $\hat{\psi}_i$ are defined by expressions (17). We will take into account such a conjugate endomorphism in the future.

If the algebra $\mathcal{A}$ has a trivial center $\mathcal{Z}(\mathcal{A}) = CI$, then the category $\mathbf{Rep(G)}$ can be embedded as a complete subcategory in the category $\mathbf{end}(\mathcal{A})$. At the same time, as it was shown in [22], those representations that satisfy the DHR criterion form a symmetric tensor C^*-category $\mathbf{rep}(\mathcal{A})$, isomorphic to $\mathbf{Rep(G)}$. The algebras of observables that satisfy the basic physical assumptions (axioms) [19, 20], have a trivial center. Therefore, the pair $(\mathcal{A}, \mathcal{T})$ corresponds to the superselection structure of the algebra $\mathcal{A}$ and mathematically generalizes in an abstract sense the concrete Tannaka–Krein duality (where the dual object of a compact group G is the category of its representations $\mathbf{Rep(G)}$). This generalization called the Doplicher–Roberts duality.

3. MODEL OF A THREE-LEVEL SYSTEM

3.1. Superselection Sectors of the Three-Level System

In the case of r particles with non-Abelian charges, superselection sectors are identified with orthogonal coherent subspaces of decomposition into the direct sum of the tensor product $\mathcal{H}^r$ (where by $\mathcal{H}$ we will further mean $\mathcal{H}_\rho$). Representation of π^r, which acts in $\mathcal{H}^r$, is also reducible and decomposes into a direct sum of irreducible representations (superselection sectors) acting on proper coherent subspaces.

In the case of a three-level system (qutrit), the state space is a three-dimensional Hilbert space $\mathcal{H}$ (where $d = 3$) formed by a linear span $\mathcal{H} = \mathrm{Lin}\{\psi_i\}_{i=1}^3$ of the multiplet ψ_1, ψ_2, ψ_3. This multiplet forms an orthonormal basis in the space $\mathcal{H}$, where the fundamental representation π of the group $SU(3)$ is realized. The scalar product is defined by (3).

Multiplet $\{\psi_i\}_{i=1}^3$ generates $*$-algebra ${}^0\mathcal{O}_3$, completion which by C^*-norm forms the Cuntz algebra $\mathcal{O}_3$. Its subalgebra $O_{G=SU(3)}$, which is pointwise fixed with respect to the action of the group $SU(3)$, is associated with the algebra of observables. $\mathcal{O}_{SU(d)}$ is generated as C^*-algebra by (6) and (7).

[1)]Here and in the future we will use the notation of clause 2.1 without corresponding references.

3.2. The Haar Measure of the SU(3) Group

In Section 2, the importance of compact groups in duality theory was noted and it was also mentioned there that in the future the object of our attention will be the category $\mathcal{T}_\rho$ generated by one endomorphism of dimension d. Such a category is isomorphic to the category $\mathbf{Rep(G)}$ for some compact Lie group G (see Theorem 4.5 in monograph [19]). This group also plays a major role in the study of the superselection structure of the C^*-algebra ${}^0\mathcal{O}_G$. Since our further presentation is also related to Lie groups (in particular, we will average (over a group of $SU(3)$) projectors on the basis states of coherent orthogonal subspaces of three-level systems), in order to facilitate the reading of the article, we will send preliminary information in a concise form without strict mathematical definitions. For more detailed information, you can refer, for example, to the classical literature [29–31].

Lie groups are isomorphic to linear groups of non-degenerate matrices of a given dimension that are subgroups of the group $GL(n)$ over the field of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers.

In the future we will focus on compact connected Lie groups, i.e., when the range of variation of their parameters is limited and includes all their limit values (compactness) and any closed path in G can be pulled to a point (simply connected)[2)]. If any two elements of a group can be translated into each other by continuously changing parameters, then such a group is not connected. In the case of connected Lie groups, almost all information about the group is contained in the tangent space to the mentioned surface at the point of the unit element. An anticommutative bilinear operation satisfying the Jacobi identity is defined in the tangent space. With respect to this operation (multiplication), the tangent space forms an algebra — Lie algebra. The connection between the algebra and the Lie group is carried out through the exponentiating.

So, the subject of our attention in this paper is the compact simply connected Lie group $G = SU(3)$. The unitary unimodular group $SU(3)$ is an eight-parameter group. In the physical literature, the Lie algebra $\mathfrak{su}(3)$ of the group $SU(3)$ is usually considered the space of Hermitian matrices with zero trace. The generators of this algebra are matrices

$$\lambda_1 = \left(\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right), \quad \lambda_2 = \left(\begin{array}{ccc} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{array}\right), \quad \lambda_3 = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right),$$

$$\lambda_4 = \left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right), \quad \lambda_5 = \left(\begin{array}{ccc} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{array}\right), \quad \lambda_6 = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right),$$

$$\lambda_7 = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{array}\right), \quad \lambda_8 = \frac{1}{\sqrt{3}}\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{array}\right), \tag{9}$$

and moreover, an arbitrary element of the group $SU(3)$ can be represented as

$$U = \exp\left(i\sum_k^8 \alpha_k\lambda_k\right), \tag{10}$$

where λ_k are Gell–Mann matrices, α_k are real coefficients.

[2)]In physics, non-compact Lie groups arise when considering space-time symmetries and the dimension of such groups is restricted. Groups of internal symmetries — groups of transformations of abstract spaces are compact and generally speaking, there are no restrictions on their dimension (i.e., on the number of parameters). According to modern concepts, these groups act independently: transformations performed in space-time manifolds do not cause changes in the internal abstract spaces.

In [32], parametrization of matrix elements of the SU(3) group using generalized Euler angles is given, and expression (10), taking into account expressions (9), can be written as

$$U = e^{i\lambda_3\alpha_1} e^{i\lambda_2\alpha_2} e^{i\lambda_3\alpha_3} e^{i\lambda_5\alpha_4} e^{i\lambda_3\alpha_5} e^{i\lambda_2\alpha_6} e^{i\lambda_3\alpha_7} e^{i\lambda_8\alpha_8}, \tag{11}$$

where

$$0 \le \alpha_1, \alpha_3, \alpha_5, \alpha_7, \le \pi; \quad 0 \le \alpha_2, \alpha_4, \alpha_6 \le \pi/2; \quad 0 \le \alpha_8 \le \pi/\sqrt{3}. \tag{12}$$

According to [32], the invariant normalized Haar measure is represented as

$$d\mu(g) = \frac{4\sqrt{3}}{\pi^5} \sin 2\alpha_2 \cos\alpha_4 \sin^3\alpha_4 \sin 2\alpha_6 d\alpha_1 d\alpha_2 \cdot \ldots \cdot d\alpha_8. \tag{13}$$

Considering that the multipliers in (11) have the form

$$e^{i\lambda_3\alpha_1} = \begin{pmatrix} ccce^{i\alpha_1} & 0 & 0 \\ 0 & e^{-i\alpha_1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e^{i\lambda_2\alpha_2} = \begin{pmatrix} ccc\cos\alpha_2 & \sin\alpha_2 & 0 \\ -\sin\alpha_2 & \cos\alpha_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$e^{i\lambda_3\alpha_3} = \begin{pmatrix} ccce^{i\alpha_3} & 0 & 0 \\ 0 & e^{-i\alpha_3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e^{i\lambda_5\alpha_4} = \begin{pmatrix} ccc\cos\alpha_4 & 0 & i\sin\alpha_4 \\ 0 & 1 & 0 \\ i\sin\alpha_4 & 0 & \cos\alpha_4 \end{pmatrix},$$

$$e^{i\lambda_3\alpha_5} = \begin{pmatrix} ccce^{i\alpha_5} & 0 & 0 \\ 0 & e^{-i\alpha_5} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e^{i\lambda_2\alpha_6} = \begin{pmatrix} ccc\cos\alpha_6 & \sin\alpha_6 & 0 \\ -\sin\alpha_6 & \cos\alpha_6 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$e^{i\lambda_3\alpha_7} = \begin{pmatrix} ccce^{i\alpha_7} & 0 & 0 \\ 0 & e^{-i\alpha_7} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e^{i\lambda_8\alpha_8} = \begin{pmatrix} ccce^{i\frac{\alpha_8}{\sqrt{3}}} & 0 & 0 \\ 0 & e^{i\frac{\alpha_8}{\sqrt{3}}} & 0 \\ 0 & 0 & e^{-i\frac{2\alpha_8}{\sqrt{3}}} \end{pmatrix}, \tag{14}$$

we obtain the following matrix elements of the matrix (10)

$$\alpha_{11} = e^{i\alpha_1} e^{i\alpha_3} e^{i\alpha_5} e^{i\alpha_7} e^{i\frac{\alpha_8}{\sqrt{3}}} \cos\alpha_2 \cos\alpha_4 \cos\alpha_6$$
$$- e^{i\alpha_1} e^{-i\alpha_3} e^{i\alpha_5} e^{i\alpha_7} e^{i\frac{\alpha_8}{\sqrt{3}}} \sin\alpha_2 \sin\alpha_6;$$

$$\alpha_{12} = e^{i\alpha_1} e^{i\alpha_3} e^{i\alpha_5} e^{-i\alpha_7} e^{i\frac{\alpha_8}{\sqrt{3}}} \cos\alpha_2 \cos\alpha_4 \sin\alpha_6$$
$$+ e^{i\alpha_1} e^{-i\alpha_3} e^{-i\alpha_5} e^{-i\alpha_7} e^{i\frac{\alpha_8}{\sqrt{3}}} \sin\alpha_2 \cos\alpha_6;$$

$$\alpha_{13} = e^{i\alpha_1} e^{i\alpha_3} e^{-i\frac{2\alpha_8}{\sqrt{3}}} \cos\alpha_2 \sin\alpha_4;$$

$$\alpha_{21} = -e^{-i\alpha_1} e^{i\alpha_3} e^{i\alpha_5} e^{i\alpha_7} e^{i\frac{\alpha_8}{\sqrt{3}}} \sin\alpha_2 \cos\alpha_4 \cos\alpha_6$$
$$- e^{-i\alpha_1} e^{-i\alpha_3} e^{i\alpha_5} e^{i\alpha_7} e^{i\frac{\alpha_8}{\sqrt{3}}} \cos\alpha_2 \sin\alpha_6;$$

$$\alpha_{22} = -e^{-i\alpha_1} e^{i\alpha_3} e^{i\alpha_5} e^{-i\alpha_7} e^{i\frac{\alpha_8}{\sqrt{3}}} \sin\alpha_2 \cos\alpha_4 \sin\alpha_6 + e^{-i\alpha_1} e^{-i\alpha_3} e^{-i\alpha_5} e^{-i\alpha_7} e^{i\frac{\alpha_8}{\sqrt{3}}} \cos\alpha_2 \cos\alpha_6;$$

$$\alpha_{23} = -e^{-i\alpha_1} e^{i\alpha_3} e^{-i\frac{2\alpha_8}{\sqrt{3}}} \sin\alpha_2 \sin\alpha_4; \quad \alpha_{31} = ie^{i\alpha_5} e^{i\alpha_7} e^{i\frac{\alpha_8}{\sqrt{3}}} \sin\alpha_4 \cos\alpha_6;$$

$$\alpha_{32} = ie^{i\alpha_5} e^{-i\alpha_7} e^{i\frac{\alpha_8}{\sqrt{3}}} \sin\alpha_4 \sin\alpha_6; \quad \alpha_{33} = e^{-i\frac{2\alpha_8}{\sqrt{3}}} \cos\alpha_4. \tag{15}$$

Also note that in the case of diagonal matrices λ_3 and λ_8, the equalities in (14) is obvious, and in the case of non-diagonal matrices λ_2 and λ_5 we used the relation

$$\exp(i\vartheta H) = I + iH\sin\vartheta + H^2(\cos\vartheta - 1), \tag{16}$$

which is valid for arbitrary Hermitian traceless matrices H with det $H = 0$. This the relation is also valid for the first seven Gell–Mann matrices. It is noted in [33] that formula (16) coincides with the Euler–Rodriguez formula for the group $SO(3)$ of rotations around the axis $\vec{n}$ generated by spin matrices $H = \vec{n}\cdot\vec{J}\,(J = 3)$ and hence $SO(3) \subset SU(3)$.

3.3. Representation Space

Here we restrict ourselves to the case of two qutrites whose superselection sectors correspond to the product of two endomorphisms with dimension $\dim(\rho) = 3$. The product $\rho\rho$ in this case can be decomposed into a direct sum of $\rho\rho = \rho_6 \oplus \bar{\rho}_3$ endomorphisms with $\dim(\rho_6) = 6$ and $\dim(\bar{\rho}_3) = 3$, where $\bar{\rho}_3$ — conjugate endomorphism. However, in the future we will use the $\rho \to \mathcal{H}_\rho$ correspondence and work with the more familiar tools of Hilbert spaces. Otherwise, we would have to deal with more abstract concepts of categorical formalism (concepts of subobject, determinant, etc.).

So we have two coherent orthogonal subspaces $\mathcal{H}\otimes\mathcal{H} = \mathcal{H}_6 \oplus \bar{\mathcal{H}_3}$, in each of which the irreducible representations π_6 and $\bar{\pi}_3$ of the group $SU(3)$ act, where $\pi\otimes\pi = \pi_6 \oplus \bar{\pi}_3$.

The basis of the space $\bar{\mathcal{H}_3}$, in which the conjugate representation of the group $SU(3)$ acts, is defined by antisymmetric tensors [17]

$$\left.\begin{aligned}\hat{\psi}_1 &= \frac{1}{\sqrt{2}}(\psi_2\psi_3 - \psi_3\psi_2);\\ \hat{\psi}_2 &= \frac{1}{\sqrt{2}}(\psi_3\psi_1 - \psi_1\psi_3);\\ \hat{\psi}_3 &= \frac{1}{\sqrt{2}}(\psi_1\psi_2 - \psi_2\psi_1).\end{aligned}\right\} \tag{17}$$

The basis of the space $\mathcal{H}_6$ in this case are symmetric tensors

$$\left.\begin{aligned}e_1 &= \psi_1\psi_1;\\ e_2 &= \frac{1}{\sqrt{2}}(\psi_1\psi_2 + \psi_2\psi_1);\\ e_3 &= \frac{1}{\sqrt{2}}(\psi_1\psi_3 + \psi_3\psi_1);\\ e_4 &= \psi_2\psi_2;\\ e_5 = \frac{1}{\sqrt{2}}(\psi_2\psi_3 + \psi_3\psi_2);\ e_6 &= \psi_3\psi_3.\end{aligned}\right\} \tag{18}$$

Here ψ_1, ψ_2, ψ_3 form the basis of the space $\mathcal{H}$ and satisfy the Cuntz relations (1) and (2). Thus, the 9-dimensional state space is divided into two superselection sectors, one of which is a conjugate sector. Projectors to the basic states of the $\mathcal{H}_6$ space are defined by expressions (by (using (18))

$$\begin{gathered}\Pi_{11} = e_1e_1^* \equiv \psi_{11}\psi_{11}^*, \quad \Pi_{12} = e_2e_2^* \equiv \psi_{12}\psi_{12}^*, \quad \Pi_{13} = e_3e_3^* \equiv \psi_{13}\psi_{13}^*,\\ \Pi_{22} = e_4e_4^* \equiv \psi_{22}\psi_{22}^*, \quad \Pi_{23} = e_5e_5^* \equiv \psi_{23}\psi_{23}^*, \quad \Pi_{33} = e_6e_6^* \equiv \psi_{33}\psi_{33}^*.\end{gathered} \tag{19}$$

Similarly, taking into account (17), for projectors on the basic states of the space $\bar{\mathcal{H}_3}$, we obtain the expressions

$$\hat{\Pi}_{23} = \hat{\psi}_1\hat{\psi}_1^* \equiv \hat{\psi}_{23}\hat{\psi}_{23}^*, \quad \hat{\Pi}_{31} = \hat{\psi}_2\hat{\psi}_2^* \equiv \hat{\psi}_{31}\hat{\psi}_{31}^*, \quad \hat{\Pi}_{12} = \hat{\psi}_3\hat{\psi}_3^* \equiv \hat{\psi}_{12}\hat{\psi}_{12}^*. \tag{20}$$

3.4. The Transfer of Quantum Information

As an illustration, consider the transfer of quantum information using two qutrites. At the same time, we will call the sender Alice, and the recipient Bob, as standard. For example, using the first of the relations (19) and (15), the state (pure) ψ_{11} in the rotated coordinate system, we define as [13]

$$\pi_6(g)\psi_{11} = \pi_6(g)\psi_1\psi_1 = \pi_6(g)\psi_1\pi_6(g)\psi_1$$
$$= \alpha_{11}^2\psi_{11} + \alpha_{11}\alpha_{21}\sqrt{2}\psi_{12} + \alpha_{11}\alpha_{31}\sqrt{2}\psi_{13} + \alpha_{21}^2\psi_{22} + \alpha_{21}\alpha_{31}\sqrt{2}\psi_{23} + \alpha_{31}^2\psi_{33}. \tag{21}$$

Similarly, for conjugate basis elements $\psi_1^*, \psi_2^*, \psi_3^*$ we get

$$\psi_{11}^*\pi_6^+(g) = \psi_1^*\psi_1^*\pi_6^+(g)$$
$$+ \bar{\alpha}_{11}^2\psi_{11}^* + \bar{\alpha}_{11}\bar{\alpha}_{21}\sqrt{2}\psi_{12}^* + \bar{\alpha}_{11}\bar{\alpha}_{31}\sqrt{2}\psi_{13}^* + \bar{\alpha}_{21}^2\psi_{22}^* + \bar{\alpha}_{21}\bar{\alpha}_{31}\sqrt{2}\psi_{23}^* + \bar{\alpha}_{31}^2\psi_{33}^*. \tag{22}$$

The projector Π_{11} to the pure state of ψ_{11} is transformed by the action of the group as

$$\pi_6(g)\Pi_{11}\pi_6^+(g) = \pi_6(g)\psi_{11}\psi_{11}^*\pi_6^+(g). \tag{23}$$

Therefore, considering expressions (21) and (22), we obtain

$$\pi_6(g)\Pi_{11}\pi_6^+(g) = (\alpha_{11}\bar{\alpha}_{11})^2\Pi_{11} + 2\alpha_{11}\bar{\alpha}_{11}\alpha_{21}\bar{\alpha}_{21}\Pi_{12} + 2\alpha_{11}\bar{\alpha}_{11}\alpha_{31}\bar{\alpha}_{31}\Pi_{13}$$
$$+ (\alpha_{21}\bar{\alpha}_{21})^2\Pi_{22} + 2\alpha_{21}\bar{\alpha}_{21}\alpha_{31}\bar{\alpha}_{31}\Pi_{23} + (\alpha_{31}\bar{\alpha}_{31})^2\Pi_{33} + \alpha_{11}^2\bar{\alpha}_{11}\bar{\alpha}_{21}\sqrt{2}\psi_{11}\psi_{12}^*$$
$$+ \alpha_{11}^2\bar{\alpha}_{11}\bar{\alpha}_{31}\sqrt{2}\psi_{11}\psi_{13}^* + \alpha_{11}^2\bar{\alpha}_{21}^2\psi_{11}\psi_{22}^* + \alpha_{11}^2\bar{\alpha}_{21}\bar{\alpha}_{31}\sqrt{2}\psi_{11}\psi_{23}^* + \ldots \tag{24}$$

Expression (24) contains a total of 36 terms, among which there are 6 projectors to the basic states (18) and 30 operators corresponding to transitions between the states of the basis with coefficients depending on the matrix elements (15) of the matrix of the fundamental representation of the group $SU(3)$.

The group averaging procedure requires the calculation of the integral (see (23))

$$\tilde{\Pi}_{11} = \int\limits_G \pi_6(g)\Pi_{11}\pi_6^+(g)d\mu(g), \tag{25}$$

where the Haar measure is defined using the expressions (12), (13). Substituting expression (24) into (25), we see that among 36 integrals, 30 will be zero due to the appearance of integrals of the type $\int_0^\pi \exp(i2\alpha_1)d\alpha_1 = 0$, etc. for products of $\alpha_{11}\bar{\alpha}_{21}, \ldots$ matrix elements (15). As a result, integrating the coefficients in (24) with projectors on the basic states of the sector gives

$$\tilde{\Pi}_{11} = \frac{1}{6}\left(\Pi_{11} + \Pi_{12} + \Pi_{13} + \Pi_{23} + \Pi_{22} + \Pi_{33}\right). \tag{26}$$

Since projectors (19) and (20) depend on the coordinate system, the pure state (for example, Π_{11}) prepared by Alice in her coordinate system, Bob must average when receiving (in the absence of correlation between their systems), perceiving it as a mixed state (26). It is also easy to make sure that $[\tilde{\Pi}_{11}, G] = 0$, i.e., $\tilde{\Pi}_{11}$ belongs to the algebra of observables $\mathcal{O}_G$.

Similarly, it can be shown that averaging over a group of any pure state from the sector $\bar{\mathcal{H}}_3$, for example, $\hat{\Pi}_{23}$, leads to a mixed state

$$\tilde{\hat{\Pi}}_{23} = \frac{1}{3}\left(\hat{\Pi}_{23} + \hat{\Pi}_{31} + \hat{\Pi}_{12}\right). \tag{27}$$

The information encoded by Alice, using the pure states of the sectors $\mathcal{H}_6$ and $\bar{\mathcal{H}}_3$, taking into account (27), is recognized by Bob by a projective measurement performed using the superselection operator (8)

$$\mathfrak{S} = \mu_1\left(\frac{1}{6}(\Pi_{11} + \Pi_{12} + \Pi_{13} + \Pi_{23} + \Pi_{22} + \Pi_{33})\right) + \mu_2\left(\frac{1}{3}(\hat{\Pi}_{12} + \hat{\Pi}_{23} + \hat{\Pi}_{31})\right), \tag{28}$$

where μ_1 and μ_2 are probabilities determined by the frequency of the sent signals encoded respectively by symmetric and antisymmetric states of the sectors $\mathcal{H}_6$, $\bar{\mathcal{H}}_3$.

3.5. Algebra of Observables and Sector Statistics

Algebra of observables

According to (5), consider the $*$-algebra ${}^0\mathcal{O}_3 = \oplus_k^0 \mathcal{O}_3^k$, where $k = 0, \pm 1, \pm 2, \pm 3$, $r = 0, 1, 2, 3$, i.e., the algebraic part of the C^*-Cuntz algebra $\mathcal{O}_3$. The generators of this algebra are finite linear combinations of maps of the form (4), and, therefore, the structures of the operators of the corresponding spaces ${}^0\mathcal{O}_3^k$ for various values of k are determined using these expressions, taking into account the structure of the bases of the resulting orthogonal coherent subspaces. For example, consider an operator from ${}^0\mathcal{O}_3^0$ with $r = 2$ of the form $t \in (\mathcal{H}^{\otimes 2}, \mathcal{H}^{\otimes 2})$, which is the direct sum of the operators t_1, t_2, t_3 and t_4 due to the decomposition of $\mathcal{H} \otimes \mathcal{H} = \mathcal{H}_6 \oplus \bar{\mathcal{H}}_3$ with basis defined by (17) and (18). Considering the expression (4), as well as (17) and (18), we have $t_1 = \hat{\psi}_i \hat{\psi}_j^* \in (\bar{\mathcal{H}}_3, \bar{\mathcal{H}}_3)$ $(i, j = 1, 2, 3)$, $t_2 = e_i \hat{\psi}_j^* \in (\bar{\mathcal{H}}_3, \mathcal{H}_6)$ $(i = 1, 2, ..., 6; j = 1, 2, 3)$, $t_3 = \hat{\psi}_i e_j^* \in (\mathcal{H}_6, \bar{\mathcal{H}}_3)$ $(i = 1, 2, 3; j = 1, 2, ..., 6)$, $t_4 = e_i e_j^* \in (\mathcal{H}_6, \mathcal{H}_6)$ $(i, j = 1, 2, ..., 6)$. The operators t_1 and t_4 are linear transformations of the spaces $\bar{\mathcal{H}}_3$ and $\mathcal{H}_6$ into themselves and simultaneously intertwining operators of the corresponding endomorphisms of the category $\mathcal{T}_\rho$, i.e., $t_1 \in (\bar{\rho}_3, \bar{\rho}_3)$, $t_4 \in (\rho_6, \rho_6)$. In addition, they are projectors on the basis vectors of the spaces $\bar{\mathcal{H}}_3$ and $\mathcal{H}_6$. Thus, these operators act inside the sector without changing the number of qutrites and therefore belong to the algebra of observables $\mathcal{O}_{SU(3)}$. Projectors (19) and (20) are obtained from t_1 and t_4 for i=j. Operators $t_2 \in (\bar{\rho}_3, \rho_6)$ and $t_3 \in (\rho_6, \bar{\rho}_3)$ act between sectors and play the role of field operators that change the charge of the sector.

Similarly, operators from ${}^0\mathcal{O}_3^1$, ${}^0\mathcal{O}_3^2$, etc. can be analyzed. Thus, in the presence of a compact group $SU(3)$, all operators belonging to $\mathcal{O}_3$ are divided into two classes of operators: acting within sectors and between sectors. The operators acting inside the sectors belong to the subalgebra $\mathcal{O}_{SU(3)} \subset \mathcal{O}_3$, which corresponds to the algebra of observables within the framework of the model. The operators acting between sectors change the charge number of the sector and belong to the field algebra $\mathcal{O}_3$.

In addition, there are unitary operators $\vartheta(r, s) : \mathcal{H}^r \otimes \mathcal{H}^s \to \mathcal{H}^s \otimes \mathcal{H}^r$, also belonging to the algebra of observables, i.e., $\vartheta(r, s) \in (\mathcal{H}^{r+s}, \mathcal{H}^{r+s})_G$ [25]. In the case of the above orthogonal subspaces we have

$$\vartheta(\bar{\mathcal{H}}_3, \bar{\mathcal{H}}_3) = \sum_{i=1}^{3} \sum_{j=1}^{3} \hat{\psi}_i \hat{\psi}_j \hat{\psi}_i^* \hat{\psi}_j^*, \tag{29}$$

$$\vartheta(\bar{\mathcal{H}}_3, \mathcal{H}_6) = \sum_{i=1}^{6} \sum_{j=1}^{3} e_i \hat{\psi}_j e_i^* \hat{\psi}_j^*, \tag{30}$$

and

$$\vartheta(\mathcal{H}_6, \mathcal{H}_6) = \sum_{i=}^{6} \sum_{j=1}^{6} e_i e_j e_i^* e_j^*. \tag{31}$$

As we will see below, the operators (29)–(31) play an important role in determining sector statistics.

Statistics of Superselection Sectors

In quantum mechanics, linear combinations of solutions of the Schrodinger equation for a system of n identical particles, differing only in permutations of their coordinates, are divided by the type of symmetry into various non-miscible combinations, forming bases of irreducible representations of the permutation group. According to the exclusion principle, a system of identical particles having spin s can only be in symmetric or antisymmetric states that are not degenerate by permutations, since the total wave function of such a system is multiplied by $(-1)^{2s}$ when any pair of particles is permuted. Therefore, states can be classified according to their statistics[3)].

[3)]Note that already within the framework of standard quantum mechanics, the existence of so-called paroparticles is allowed, which can be described using multidimensional representations of the group $\mathbf{P}_n$. Theoretically, paroparticles can include real particles that differ in certain internal quantum numbers (for example, color, isospin), as well as some quasiparticles, such as Frenkel excitons, magnons in periodic lattices.

However, in such systems, where the interpretation in terms of particles becomes practically meaningless (for example, when it is necessary to take into account the interaction), the statistics of the system cannot be determined using permutations of particles. One of the approaches to solving this problem has been developed in algebraic quantum field theory [19]. Its essence is in considering the statistics of the superselection sector, which is determined solely on the basis of the observed values belonging to this sector. Using the left inverse ϕ (see Appendix) to each superselection sector, a number (statistical parameter) $\lambda_\rho = \phi(\varepsilon(\rho,\rho))$ can be compared in the only way, which marks the irreducible representations of the permutation group $\mathbf{P}_n$. Here $\varepsilon(\rho,\rho) \in (\rho^2,\rho^2)$ is a unitary operator in the symmetric tensor category (see Appendix). For an irreducible endomorphism ρ we have that $\phi(\varepsilon(\rho,\rho)) = \lambda_\rho \mathbb{I}$, where $\lambda_\rho \in \{0\} \cup \{\pm d^{-1} : d \in \mathbb{N}\}$, and $\lambda_\rho = \frac{1}{d}$ corresponds to a parabose statistics of the order d with a Young tableau having a column length of $\leq d$, $\lambda_\rho = -\frac{1}{d}$ corresponds to parafermi statistics of the order of d and a Young tableau having a string length of $\leq d$. The case $\lambda_\rho = 0$ describes infinite statistics that are not observed for real particles[4)]. Ordinary statisticians Bose and Fermi correspond to the values of $\lambda_\rho = \pm 1$. The last two simple cases are realized if only ρ is an automorphism of the algebra of observables. The natural number d is called the statistical dimension of a superselection sector, which coincides with the notion of the dimension $\dim(\rho)$ of an object ρ of a symmetric tensor category. In the case of the Cuntz algebra, we have $\phi(c) = (1/d)\sum_{i=1}^{d} \psi_i^* c \psi_i$, $c \in \mathcal{O}_d$ [25]. In addition, in our model, the equality $\varepsilon(\rho,\rho) = \vartheta(\rho,\rho) = \vartheta(\mathcal{H},\mathcal{H})$ is valid, therefore, for the sectors discussed above, we have

$$\phi(\varepsilon(\bar{\rho}_3,\bar{\rho}_3)) = \phi(\vartheta(\bar{\mathcal{H}}_3,\bar{\mathcal{H}}_3)) = \frac{1}{3}\sum_{k,i,j} \hat{\psi}_k^* \hat{\psi}_i \hat{\psi}_j \hat{\psi}_i^* \hat{\psi}_j^* \hat{\psi}_k,$$

where $\vartheta(\bar{\mathcal{H}}_3,\bar{\mathcal{H}}_3)$ is defined by expression (29). Since $\hat{\psi}_k^*\hat{\psi}_i = \delta_{ki}$ and $\hat{\psi}_j^*\hat{\psi}_k = \delta_{jk}$ according to the Cuntz relations (1) and (2), then $\phi(\vartheta(\bar{\mathcal{H}}_3,\bar{\mathcal{H}}_3)) = 1/3\sum_k \hat{\psi}_k\hat{\psi}_k^* = (1/3)1_{\bar{\mathcal{H}}_3}$, $1_{\bar{\mathcal{H}}_3} \in (\bar{\mathcal{H}}_3,\bar{\mathcal{H}}_3)$ (more details about conjugate objects are given in [17]). Therefore, the statistical parameter $\lambda = 1/3$ defines a sector with parabose statistics of order 3. Similarly,

$$\phi(\varepsilon(\rho_6,\rho_6)) = \phi(\vartheta(\mathcal{H}_6,\mathcal{H}_6)) = 1/6\sum_{i,j,k=1}^{6} e_k^* e_i e_j e_i^* e_j^* e_k = (1/6)1_{\mathcal{H}_6},$$

and we get a sector with parabose statistics of order 6 (where $1_{\mathcal{H}_6} \in (\mathcal{H}_6,\mathcal{H}_6)$ and $\vartheta(\mathcal{H}_6,\mathcal{H}_6)$ is defined by the expression (31)).

4. CONCLUSIONS

In this paper, the algebraic model proposed by us in [13] is extended, taking into account the conjugate endomorphism with dimension $\dim(\rho) = 3$. The basis of the model is symmetric tensor C^*-category whose objects are non-Abelian superselection sectors of the physical system under consideration [22] (the role of category theory in modern quantum theory can be found, for example, in [34, 35]). Arrows of a category that are intertwining operators and change the charge of a given sector can be considered as elements of a field algebra, and morphisms acting inside a sector can be interpreted as elements of the algebra of observables. Taking into account the conjugate object therefore allows you to study both the operators that translate the conjugate sector into themselves, and the conjugation operators that make the transition from the usual sector to the conjugate, which we considered in clause 3.5. As an approbation, the process of transmitting quantum information using a three-level system was also considered, and it was seen that the amount of information transmitted is equal to the number of superselection sectors. Here we have restricted ourselves only to the case of decomposition of two qutrites into two coherent superselection sectors (defined by (28)): a three-dimensional conjugate sector and a six-dimensional ordinary sector ($\mathcal{H}\otimes\mathcal{H} = \mathcal{H}_6 \oplus \bar{\mathcal{H}}_3$). Other possible decomposition channels remained outside our attention (for example, in the case of two qutrites, the case of $\bar{\mathcal{H}}\otimes\bar{\mathcal{H}} = \iota \oplus ...$, where ι is a vacuum non-Abelian sector) and analysis of the connection of the (possible) degree of security of cryptographic protocols with conjugated non-Abelian charges.

[4)]This case describes anions.

Appendix

A category $\mathcal{C}$ is called a C^*-category if the set of morphisms (ρ, ρ_1) between two objects ρ, ρ_1 forms a complex Banach space and the composition of morphisms is a bilinear map of $t, s \to t \circ s$ with $||t \circ s|| \leq ||t|| \circ ||s||$. In this category, there is a contravariant functor $*$ that reverses morphisms and acts identically on objects, and the norm of the morphism satisfies C^* is the property $||r^* \circ r|| = ||r||^2$ for any $r \in (\rho, \rho_1)$. The set of morphisms (ρ, ρ) in C^*-categories $\mathcal{C}$ forms C^*-algebra for each $\rho \in \mathbf{obj}\,\mathcal{C}$.

Tensor C^*-category $\mathcal{C}$ is a C^*-category equipped with a tensor product $\otimes$. This means that each pair of objects ρ, ρ_1 corresponds to an object $\rho \otimes \rho_1$, and $\mathcal{C}$ has an identical object ι, for which the relation $\rho \otimes \iota = \rho = \iota \otimes \rho$. Moreover, for two morphisms $t \in (\rho, \rho_1)$ and $s \in (\rho_2, \rho_3)$ there is a morphism $t \times s \in (\rho \otimes \rho_2, \rho_1 \otimes \rho_3)$. For the case of the category of endomorphisms of the algebra that we used in this paper, the relation holds

$$t \times s = t\rho(s) = \rho_1(s)t. \tag{32}$$

The mapping $t, s \to t \times s$ is associative and bilinear and

$$1_\iota \times t = t = t \times 1_\iota, \quad (t \times s)^* = t^* \times s^*.$$

Interchange law holds

$$t \times s \circ t_1 \times s_1 = (t \circ t_1) \times (s \circ s_1), \tag{33}$$

whenever the left hand side is defined (note that $\times$ will be evaluated before $\circ$).

Such categories are often called strict monoidal, since the associativity law

$$(\rho \otimes \rho_1) \otimes \rho_2 = \rho \otimes (\rho_1 \otimes \rho_2), \quad \iota \otimes \rho = \rho \otimes \iota = \rho$$

is strictly executed. A similar relation holds for morphisms. In other words, a strict monoidal category is a C^*-category where associative bilinear functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ commuting with the conjugation operation $*$. Also note that in a strict monoidal category, the set of morphisms (ρ, ρ_1) forms not only the structure of a vector space, but also has the natural structure of a (ι, ι)-bimodule.

The category $\mathcal{C}$ is called closed with respect to subobjects if for each projector $E \in (\rho, \rho)$ there is an isometry $t \in (\rho_1, \rho)$ such that $tt^* = E$. The category $\mathcal{C}$ is called closed with respect to direct sums if for the given objects ρ_i $(i = 1, 2)$ there exists an object ρ and isometries $s_i \in (\rho_i, \rho)$ such that $s_1 s_1^* + s_2 s_2^* = 1_\rho$.

The category $\mathcal{C}$ is called *symmetric* if there is permutation symmetry, i.e., if there is a mapping $\varepsilon : \mathcal{C} \ni \rho_1, \rho_2 \longrightarrow \varepsilon(\rho_1, \rho_2) \in (\rho_1 \otimes \rho_2, \rho_2 \otimes \rho_1)$ that satisfies the conditions

$$\begin{aligned}
&(1)\ \ \varepsilon(\rho_3, \rho_4) \circ s \times t = t \times s \circ \varepsilon(\rho_1, \rho_2),\\
&(2)\ \ \varepsilon(\rho_1, \rho_2)^* = \varepsilon(\rho_2, \rho_1),\\
&(3)\ \ \varepsilon(\rho_1, \rho_2 \otimes \rho) = 1_{\rho_2} \times \varepsilon(\rho_1, \rho) \circ \varepsilon(\rho_1, \rho_2) \times 1_\rho,\\
&(4)\ \ \varepsilon(\rho_1, \rho_2) \circ \varepsilon(\rho_2, \rho_1) = 1_{\rho_2 \otimes \rho_1},
\end{aligned}$$

where $t \in (\rho_2, \rho_4), s \in (\rho_1, \rho_3)$. From (2)–(4) it follows that for any ρ the relation $\varepsilon(\rho, \iota) = \varepsilon(\iota, \rho) = 1_\rho$ is valid. Symmetric tensor categories are denoted as $(\mathcal{C}, \varepsilon)$.

Permutation symmetry for irreducible endomorphisms is conveniently classified using the left inverse. *Left inverse* of the object ρ is a set of nonzero linear maps $\phi^\rho = \{\phi^\rho_{\rho_1,\rho_2} : (\rho \otimes \rho_1, \rho \otimes \rho_2) \longrightarrow (\rho_1, \rho_2)\}$ satisfying

$$\begin{aligned}
&(1)\ \ \phi^\rho_{\rho_3,\rho_4}(1_\rho \times t \circ r \circ 1_\rho \times s^*) = t \circ \phi^\rho_{\rho_1,\rho_2}(r) \circ s^*,\\
&(2)\ \ \phi^\rho_{\rho_1 \otimes \rho_3, \rho_2 \otimes \rho_3}(r \times 1_{\rho_3}) = \phi^\rho_{\rho_1,\rho_2}(r) \times 1_{\rho_3},\\
&(3)\ \ \phi^\rho_{\rho_1,\rho_1}(s_1^* \circ s_1) \geq 0,\\
&(4)\ \ \phi^\rho_{\iota,\iota}(1_\rho) = \mathbf{1}_\iota,
\end{aligned}$$

where $s \in (\rho_1, \rho_3)$, $t \in (\rho_2, \rho_4)$, $r \in (\rho \otimes \rho_1, \rho \otimes \rho_2)$ and $s_1 \in (\rho \otimes \rho_1, \rho \otimes \rho_1)$. We say that $\mathcal{C}$ has *left inverse* if any object of this category has a left inverse. The left inverse plays an important role in describing the statistics of superselection sectors.

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