ISSN 0027-1349, Moscow University Physics Bulletin, 2023, Vol. 78, No. 4, pp. 425–433. © Allerton Press, Inc., 2023.
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THEORETICAL AND MATHEMATICAL PHYSICS

On One Conjugate Object in the Symmetric Tensor C^*-Category and the Statistics of Superselection Sectors

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Received March 15, 2023; revised April 17, 2023; accepted April 25, 2023

Abstract—The paper presents a model of symmetric tensor C^*-category with conjugation for an object of dimension $d = 3$. It is proved that the constructed conjugate object of the category satisfies the conjugation equations, and different classes of morphisms between the objects in the modelled category are considered and studied. This category allows for the generalization of the C^*-algebraic model of observables in the presence of superselection rules generated by non-Abelian conjugate charges. As an application of the model, the area of quantum information transfer is explored, where the constraints imposed by the superselection rules need to be taken into account.

Keywords: symmetric tensor C^*-category, conjugate object, superselection rules, quantum information transfer

DOI: 10.3103/S0027134923040173

INTRODUCTION

Category theory is one of the rapidly developing areas of mathematics. In recent times, particular emphasis has been placed on various branches of category theory, such as higher category theory [1], multicategory theory [2] (see also [3]), homotopy categories [4], and others. These branches hold a special status in gauge theories and algebraic quantum field theory. It is difficult to imagine logic, higher algebra, and higher geometry without category theory. The influence of category theory has also significantly impacted the development of computer science [5].

Due to the need for a rigorous description of the superselection structure within the framework of algebraic quantum field theory [6], significant progress has been made in the theory of tensor C^*-categories. The first stage of its substantial development occurred in the work [7], where a new theory of duality was formulated, generalizing the classical Tannaka–Krein duality [8, 9].

The superselection structure of the theory is closely related to internal symmetries and gives rise to dynamical superselection rules associated with absolutely conserved abelian or non-Abelian charges. Such symmetries are described by compact topological groups, which in elementary particle physics correspond to groups of global gauge transformations. The spectrum (set of unitarily nonequivalent representations) of this group forms the set of superselection sectors of the considered system. Therefore, finite-dimensional Hilbert spaces correspond to the superselection sectors, where unitarily nonequivalent representations of the compact group are realized. The authors of the work [7] were able to show that the category of finite-dimensional Hilbert spaces **hilb** is isomorphic to the abstract tensor symmetric C^*-category $\mathcal{C}$.

Non-Abelian superselection charges are associated with objects in this category, and algebraic operations of conjugation, permutation symmetry, and composition can be defined over its objects. These operations can be interpreted in physics as the transition to an antiparticle, particle statistics, and addition of charges, respectively [10].

According to the mentioned category isomorphism, the abstract category serves as the dual object to the compact group G, which is referred to as the Doplicher–Roberts duality. Using the technique of crossed products [11], the group G can be reconstructed axiomatically from the quasilocal algebra of observables $\mathcal{A}$ [6], without artificially introducing it as the group of automorphisms $aut(\mathcal{F})$ of the field

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algebra $\mathcal{F} = \mathcal{A} \times \mathcal{C}$. More detailed information on this is provided below in Subsection 1.4.

Thanks to technological advancements, quantum informatics has experienced rapid development in recent years. Modern experimental techniques enable the implementation of experiments on the transfer of quantum information encoded in the quantum states of elementary particles, atoms, and even molecules. The basic unit of quantum information in these cases is a qubit—a two-level quantum system. For instance, a photon with longitudinal and transverse polarizations, an electron with two basis states, a hydrogen molecule ion H_2^+ with basis states corresponding to electron localization around the first or second proton, an ammonia molecule NH_3 with basis states representing two mirror configurations separated by a barrier and resulting from each other by mirror reflection with respect to the plane separating these configurations, etc., are examples of two-level quantum systems that find technical applications. It has been discovered that in the case of two-level systems possessing Abelian or non-Abelian gauge charges, superselection rules arise, limiting the amount of transmitted information. This possibility was initially noted in an unpublished work by Popescu and has since been actively studied in the scientific literature. An overview of the most significant results in this field can be found in the review article [12].

The work [13] proposed the hypothesis that, in quantum information theory, conjugate charges may play a significant role alongside charges. The authors of this study investigated the role of conjugate charge in quantum cryptography and found that a conjugate abelian charge cannot enhance the security of cryptographic protocols. However, while complete annihilation with the formation of a vacuum occurs in the case of abelian charges, the situation is much more complex for non-Abelian charges. The composition of a charge with its conjugate leads not only to the formation of a vacuum sector but also to the creation of a sector containing paraparticles. Therefore, this issue is still considered open and further research is required.

In the work [14], we proposed an algebraic model to investigate the role of superselection rules in quantum information theory. This model allowed for demonstrating that information can be encoded using only those states for which projectors commute with the algebra of observables. Since these projectors also commute with representation elements of a non-abelian group, the recipient has the ability to fully recover the transmitted information. However, to study the superselection rules in the presence of a conjugate charge within this model, further refinement is required taking into account the conjugate object. Therefore, the aim of this article is to conduct preliminary work on modelling a symmetric tensor C^*-category with a conjugate object for an object dimension of $d = 3$, and to investigate certain classes of morphisms within this framework.

According to the above-mentioned outline, the article is organized as follows. Section 1 provides preliminary information on Cuntz algebras, symmetric tensor C^*-categories, and introduces some constructions that establish the connection between abstract and concrete categories within the framework of Doplicher–Roberts duality. Section 2 constitutes the original part of the work, and its main result is the construction of the conjugate object in the category with dimension $d = 3$. It is proved that the conjugation equations hold for this object. Furthermore, a left inverse mapping ϕ for the object ρ is introduced, and the statistical parameter, an invariant of the sector, is defined using it. The final subsections of the section serve as preliminary and auxiliary materials, providing additional information to the main results. Here, the algebra of observables is constructed through conditional expectation (a positive mapping from the field algebra to the algebra of observables), and a three-level quantum system is considered, which is used to encode quantum information. A more detailed analysis of information transfer using qutrits based on the results of this work is planned to be presented in a separate publication.

Finally, the conclusion provides a brief analysis of the work.

1. PRELIMINARY INFORMATION

1.1. Cuntz Algebra

The Cuntz algebra, as introduced in [15], plays a crucial role in the reconstruction of a compact group G within the Doplicher–Roberts duality theory. It also aids in establishing an isomorphism between the abstract tensor C^*-categories $\mathcal{C}$ and the category of representations $\mathbf{rep}(G)$, mentioned in the introduction. To facilitate further exposition, we will provide here the fundamental information regarding Cuntz algebras.

The Cuntz C^*-algebra is defined by the Cuntz relations

$$\psi_i^* \psi_j = \delta_{ij} I, \tag{1}$$

$$\sum_i^d \psi_i \psi_i^* = I \tag{2}$$

and by a scalar product

$$\psi^* \psi' = \langle \psi, \psi' \rangle I, \quad \psi, \psi' \in \mathcal{H}, \tag{3}$$

here, $\mathcal{H}$ is a complex Hilbert space of dimension dim $n \geq 2$ with an orthonormal basis $\{\psi_i\}_{i=1,2,...,d}$, where ψ_i are isometric operators. In other words, the mentioned d-dimensional Hilbert space is generated by the linear span $Lin\{\psi_i\}_{i=1}^d$ of the multiplet of orthogonal isometries $\{\psi\}_{i=1}^d$, and it is referred to as the canonical space.

The algebra $\mathcal{O}_d$ can be conveniently obtained using the following construction [16]. Let **hilb** be the category of tensor powers $\underbrace{\mathcal{H} \otimes ... \otimes \mathcal{H}}_{r} = \mathcal{H}^r$ of d-dimensional Hilbert spaces $\mathcal{H} = Lin\{\psi_i\}_{i=1}^d$, where $r \in \mathbb{Z}_+$ and $d > 1$. Morphisms between tensor powers from $\mathcal{H}^s$ in $\mathcal{H}^r$ are given by linear mapping of the form:

$$t = \psi_{i_1}\psi_{i_2}...\psi_{i_r}\psi^*_{j_s}...\psi^*_{j_2}\psi^*_{j_1} \in (\mathcal{H}^s, \mathcal{H}^r),$$

which form a complex Banach space (here and in the future, for brevity, the tensor product symbols between $\psi_i \otimes \psi_j$ will be omitted).

Let us define the inductive limit ${}^0O_d^k$ of injective mappings of morphisms for each fixed value $k = 0; \pm 1; \pm 2; ...$:

$$(\mathcal{H}^r, \mathcal{H}^{r+k}) \longrightarrow^{\otimes 1} (\mathcal{H}^{r+1}, \mathcal{H}^{r+1+k}) \longrightarrow^{\otimes 1} ... \\ ... \longrightarrow^{\otimes 1} (\mathcal{H}^{r+n}, \mathcal{H}^{r+n+k}) \longrightarrow^{\otimes 1} .... \quad (4)$$

Then, the algebra ${}^0\mathcal{O}_d$, obtained by taking the direct sum ${}^0\mathcal{O}_d = \oplus_k^0 \mathcal{O}_d^k$ over all values of k, is a $*$-algebra. Its completion under the unique C^*-norm yields the Cuntz algebra $\mathcal{O}_d$. In this case, the identity of the Cuntz algebra is defined according to the expression (2).

Let us consider a closed subgroup G of the group of unitary operators $U(\mathcal{H})$. We can then consider the category $\mathbf{hilb}_G$, whose objects are G-modules $\mathcal{H}^r$, and the morphisms are G-module homomorphisms. By repeating the previous construction (4)

$$(\mathcal{H}^r, \mathcal{H}^{r+k})_G \longrightarrow^{\otimes 1} (\mathcal{H}^{r+1}, \mathcal{H}^{r+1+k})_G \longrightarrow^{\otimes 1} ... \\ ... \longrightarrow^{\otimes 1} (\mathcal{H}^{r+n}, \mathcal{H}^{r+n+k})_G \longrightarrow^{\otimes 1} ..., \quad (5)$$

we obtain the $\mathbb{Z}$-graded $*$-algebra ${}^0\mathcal{O}_G = \oplus_k^0 \mathcal{O}_G^k$. According to [16], if G is a subgroup of the group $SU(\mathcal{H})$, then there exists a unique C^*-seminorm (which is, in fact, a C^*-norm) on ${}^0\mathcal{O}_G$, and $\mathcal{O}_G$ is the C^*-algebra obtained by completing ${}^0\mathcal{O}_G$ with respect to this seminorm. In this case, the algebra $\mathcal{O}_G$ is simple. The spectral subspaces of the algebra $\mathcal{O}_G$ are defined as $\mathcal{O}_G^k = \{X \in \mathcal{O}_G : \alpha_\lambda(X) = \lambda^k X\}$, where α is a continuous action of the circle group on $\mathcal{O}_G$, turning $\mathcal{O}_G$ into a $\mathbb{Z}$-graded C^*-algebra, $X \in \mathcal{O}_G$. It is worth noting that $\mathcal{O}_G^k$ are closures of the corresponding ${}^0\mathcal{O}_G^k$.

Without going into details, let us mention that in the $*$-algebras ${}^0\mathcal{O}_d$ and ${}^0\mathcal{O}_G$, it is possible to define nontrivial canonical endomorphisms σ and σ_G, respectively, which can be extended to canonical endomorphisms of the C^*-algebras $\mathcal{O}_d$ and $\mathcal{O}_G$. According to expressions (1)–(2), the canonical endomorphism is defined as $\sigma(X) = \sum_{i=1}^d \psi_i X \psi_i^*$, $X \in \mathcal{O}_d$ or $X \in \mathcal{O}_G$. If $X \in \mathcal{O}_d$, then $\sigma(X)$ is called inner.

We also note that the algebra $\mathcal{O}_G$ and its canonical endomorphism σ_G define a C^*-dynamical system

$$(\mathcal{O}_G, \sigma_G), \quad (6)$$

which plays a significant role in the Doplicher–Roberts theory.

1.2. Symmetric Tensor C^*-Categories

Let us provide some basic information about abstract symmetric tensor C^*-categories. The discussion of conjugation in such categories will be deferred until Section 3.

A category $\mathcal{C}$ is called a C^*-category if the set of morphisms (ρ, ρ_1) between two objects ρ, ρ_1 forms a complex Banach space, and composition between morphisms is a bilinear mapping $t, s \to t \circ s$ with $||t \circ s|| \leq ||t|| \circ ||s||$. In this category, there exists a contravariant functor $*$ that reverses the morphisms and acts trivially on objects. Therefore, the norm of a morphism satisfies the C^*-property $||r^* \circ r|| = ||r||^2$ for any $r \in (\rho, \rho_1)$. The set of morphisms (ρ, ρ) in the C^*-category $\mathcal{C}$ generates a C^*-algebra for each $\rho \in \mathbf{obj}\mathcal{C}$.

Tensor C^*-category $\mathcal{C}$ is a C^*-category equipped with tensor product $\otimes$. As in the case of the category **hilb**, this means that each pair of objects ρ, ρ_1 corresponds to an object $\rho \otimes \rho_1$. Additionally, $\mathcal{C}$ has an identity object (unit) ι such that $\rho \otimes \iota = \rho = \iota \otimes \rho$. Moreover, for the two morphisms $t \in (\rho, \rho_1)$ and $s \in (\rho_2, \rho_3)$, there exists a morphism $t \times s \in (\rho \otimes \rho_2, \rho_1 \otimes \rho_3)$. For the special case of the category of endomorphisms of the algebra $\mathcal{A}$, which we will use in Section 3, the relation

$$(t \times s) = t\rho(s) = \rho_1(s)t \quad (7)$$

holds. Mapping $t, s \to t \times s$ is associative and bilinear and

$$1_\iota \times t = t = t \times 1_\iota, \quad (t \times s)^* = t^* \times s^*.$$

Alternation rule

$$t \times s \circ t_1 \times s_1 = (t \circ t_1) \times (s \circ s_1) \quad (8)$$

holds, provided that the right-hand side is defined.

Such categories are often referred to as strict monoidal categories and are denoted by the triple

$(\mathcal{C}, \otimes, \iota)$, where $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is an associative bilinear functor (tensor product) that commutes with the conjugation operation $*$. In a strict monoidal category, the set of morphisms (ρ, ρ_1) not only forms the structure of a complex vector space but also has a natural (ι, ι)-bimodule structure. The category $\mathcal{C}$ is called *symmetric* if it exhibits permutation symmetry, which means that there exists a mapping $\varepsilon : \mathcal{C} \ni \rho_1, \rho_2 \longrightarrow \varepsilon(\rho_1, \rho_2) \in (\rho_1 \otimes \rho_2, \rho_2 \otimes \rho_1)$, satisfying the conditions:

(1) $\varepsilon(\rho_3, \rho_4) \circ s \times t = t \times s \circ \varepsilon(\rho_1, \rho_2)$,

(2) $\varepsilon(\rho_1, \rho_2)^* = \varepsilon(\rho_2, \rho_1)$,

(3) $\varepsilon(\rho_1, \rho_2 \otimes \rho) = 1_{\rho_2} \times \varepsilon(\rho_1, \rho) \circ \varepsilon(\rho_1, \rho_2) \times 1_\rho$,

(4) $\varepsilon(\rho_1, \rho_2) \circ \varepsilon(\rho_2, \rho_1) = 1_{\rho_2 \otimes \rho_1}$,

here $t \in (\rho_2, \rho_4), s \in (\rho_1, \rho_3)$. From 2–4, it follows that for any ρ, the relation $\varepsilon(\rho, \iota) = \varepsilon(\iota, \rho) = 1_\rho$ holds. Let us denote symmetric tensor categories as $(\mathcal{C}, \varepsilon)$.

An object ρ is called irreducible if $(\rho, \rho) = \mathbb{C}I$. The permutation symmetry for irreducible endomorphisms can be conveniently classified using the notion of a left inverse mapping [16]. Therefore, let us provide its definition. The *left inverse mapping* of an object ρ is the set of nonzero linear mappings $\phi^\rho = \{\phi^\rho_{\rho_1, \rho_2} : (\rho \otimes \rho_1, \rho \otimes \rho_2) \longrightarrow (\rho_1, \rho_2)\}$, satisfying

(1) $\phi^\rho_{\rho_3, \rho_4}(1_\rho \times t \circ r \circ 1_\rho \times s^*) = t \circ \phi^\rho_{\rho_1, \rho_2}(r) \circ s^*$,

(2) $\phi^\rho_{\rho_1 \otimes \rho_3, \rho_2 \otimes \rho_3}(r \times 1_{\rho_3}) = \phi^\rho_{\rho_1, \rho_2}(r) \times 1_{\rho_3}$,

(3) $\phi^\rho_{\rho_1, \rho_1}(s_1^* \circ s_1) \geq 0$,

(4) $\phi^\rho_{\iota, \iota}(1_\rho) = \mathbf{1}_\iota$,

here $s \in (\rho_1, \rho_3)$, $t \in (\rho_2, \rho_4)$, $r \in (\rho \otimes \rho_1, \rho \otimes \rho_2)$, and $s_1 \in (\rho \otimes \rho_1, \rho \otimes \rho_1)$. $\mathcal{C}$ is said to have a *left inverse* if every object in this category has a left inverse mapping.

1.3. Doplicher–Roberts Algebra

Let us briefly outline the scheme for constructing a C^*-algebra using the object ρ of the $\mathcal{C}$-category [7]. This algebra is commonly referred to as the Doplicher–Roberts algebra and denoted as $\mathcal{O}_\rho$. The algebra $\mathcal{O}_\rho$ is essentially isomorphic to the algebra $\mathcal{O}_G$.

Let us consider, for arbitrary $k \in \mathbb{Z}$, the Banach space $\mathcal{O}^k_\rho$ defined as the inductive limit (see also (4) and (5)):

$$... \longrightarrow^{\otimes 1} (\rho^r, \rho^{r+k}) \longrightarrow^{\otimes 1} (\rho^{r+1}, \rho^{r+k+1}) \longrightarrow^{\otimes 1} ....$$

The direct sum ${}^0\mathcal{O}_\rho = \oplus_k \mathcal{O}^k_\rho$ is a $*$-algebra, and its completion with respect to the C-norm yields the C^*-algebra $\mathcal{O}_\rho$. A detailed description of this algebra from a mathematical perspective, as well as the intricacies of its generation using the $\mathcal{C}$-category, are presented in the work [7].

1.4. Some Constructions

As mentioned in the introduction, the isomorphism between the abstract symmetric tensor C^*-categories discussed in Subsection 1.2 for the symmetric tensor C^*-categories of finite-dimensional continuous unitary representations of a compact Lie group is a consequence of the Doplicher–Roberts duality theory [7]. From a physical perspective, the category that describes the superselection structure is a symmetric tensor C^*-category of localized and transportable endomorphisms ρ of a quasilocal unital C^*-algebra $\mathcal{A}$ with an identity and a trivial center. In this category, objects are not necessarily associated with finite-dimensional Hilbert spaces, and morphisms are not associated with linear mappings between them. Such endomorphisms, for which composition, permutation symmetry, and conjugation are defined, form a semigroup. However, in the work [11], it is shown that these endomorphisms correspond to Hilbert spaces $\mathcal{H}_\rho$ in the crossed product $\mathcal{A} \times \mathcal{C}$, which can be associated with a field C^*-algebra $\mathcal{F} = \mathcal{A} \times \mathcal{C}$ in a physical sense. In these spaces, the irreducible representations of the group G are realized, and the morphisms correspond to G-module homomorphisms. The algebra $\mathcal{A}$ is a poitwise fixed subalgebra of $\mathcal{F}$ with respect to the action of the group $G \subseteq aut(\mathcal{F})$, where $aut(\mathcal{F})$ is the group of automorphisms of the algebra $\mathcal{F}$. Since the spaces $\mathcal{H}_\rho$ are generated by isometries, the Cuntz algebras $\mathcal{O}_d$ play an important role in the formulation of the Doplicher–Roberts duality, and many constructions in the abstract category $\mathcal{C}$ can be visually described using the isometries (1)–(2). It is also worth noting that the proof of the isomorphism between the C^*-dynamical system (6) and the C^*-dynamical system $(\mathcal{A}, \rho)$ in the case of $G \subseteq SU(d)$ is also a consequence of this duality [17], where the C^*-algebra $\mathcal{A}$ contains the Doplicher–Roberts algebra $\mathcal{O}_\rho$ as a subalgebra generated by the spaces of intertwining operators (ρ^r, ρ^s), $r, s \in \mathbb{N}$. In this regard, let us consider some constructions that will allow for seeing further connections between objects and morphisms of the isomorphic categories $\mathbf{hilb}_G$ and $(\mathcal{C}, \varepsilon)$.

As shown in [11, 16], the generating operators of the algebras $\mathcal{O}_G$ ($G = SU(d)$) for arbitrary d are the operators

$$\vartheta(p) = \sum_{i_1 i_2 ... i_n} \psi_{i_1} ... \psi_{i_n} \psi^*_{i_{p(n)}} ... \psi^*_{i_{p(1)}} \tag{9}$$

and

$$S = \frac{1}{\sqrt{d!}} \sum_{p \in \mathbb{P}_d} \mathrm{sgn}(p) \psi_{p(1)} ... \psi_{p(d)}. \tag{10}$$

Here $\mathbb{P}_n$ refers to the symmetric group, $p \in \mathbb{P}_n$, $\mathbb{P}_d \subset \mathbb{P}_n$.

The operator (9) is unitary and intertwines tensor powers of the form $(\mathcal{H}^r, \mathcal{H}^r)_G$ in the category $\mathbf{hilb}_G$. In particular, if $\mathcal{H}$ and $\mathcal{H}'$ are two Hilbert spaces, then

$$\vartheta(\mathcal{H}, \mathcal{H}') = \sum_{i,j} \psi'_i \psi_j \psi'^*_i \psi^*_j.$$

The image of the isometric operator S is the antisymmetric subspace in $\mathcal{H}^d = \underbrace{\mathcal{H} \otimes ... \otimes \mathcal{H}}_{d}$, where the group $SU(d)$ acts trivially on this subspace. In other words, the isometry S is invariant under the action of the group $SU(d)$: $\alpha_g(S) = S$, $g \in SU(d)$ and

$$SS^* = \frac{1}{d!} \sum_{p \in \mathbb{P}_d} \text{sgn}(p) \vartheta^d_{\mathcal{H}}(p) \tag{11}$$

determines a projector onto the completely antisymmetric subspace of the space $\mathcal{H}^d$ (where S^* is the adjoint operator). The conjugate basis can be defined as $\psi^*_i = (-1)^{d-1}\sqrt{d} S^* \hat{\psi}_i$, where

$$\hat{\psi}_i = \frac{1}{(d-1)!} \sum_{p \in \mathbb{P}_d(i)} \text{sgn}(p) \psi_{p(2)} ... \psi_{p(d)}, \tag{12}$$

here $\mathbb{P}_d(i)$ is a subset in $\mathbb{P}_d$ with $p(1) = i$.

2. CONJUGATE OBJECT AND SECTOR STATISTICS

In the case of an abelian compact group G, the superselection structure of the investigated system is determined by the discrete additive group of characters $\mathcal{X}(G)$, which is the dual object to the group G [18]. The addition of two charges ρ and ρ_1 corresponds to the composition $\rho \otimes \rho_1 \equiv \rho \circ \rho_1$. The conjugate object $\bar{\rho} \in \mathcal{X}(G)$ corresponds to an antiparticle, and the condition $\bar{\rho} \otimes \rho = \rho \otimes \bar{\rho} = \iota$ is satisfied (which physically corresponds to the annihilation of a particle—antiparticle pair). Here, ι denotes the identity element of the group. However, in the case of a non-Abelian superselection structure, the dual object of the compact group is a symmetric tensor C^*-category, and the conjugation condition cannot be interpreted in such a simple way. The key point here is the condition for the equations of conjugation to be satisfied for a morphism and its conjugate morphism, which we will demonstrate in the next section.

Below, we will consider the symmetric tensor C^*-category $\mathcal{C}_\rho$ generated by a single object ρ, and we will construct a specific conjugate object $\bar{\rho} \in \mathbf{obj}\ \mathcal{C}_\rho$. In doing so, we will use the notations and concepts introduced in the first section without providing the corresponding references.

2.1. Conjugate Object

Definition: Let $\mathcal{C}$ be a tensor C^*-category. An object $\bar{\rho} \in \mathbf{obj}\ \mathcal{C}$ is said to be conjugate to an object $\rho \in \mathbf{obj}\ \mathcal{C}$ if there exist morphisms $r : \iota \to \bar{\rho} \otimes \rho$ and $\bar{r} : \iota \to \rho \otimes \bar{\rho}$ such that the conjugation equations are satisfied for them

$$\bar{r}^* \times 1_\rho \circ 1_\rho \times r = 1_\rho;$$
$$r^* \times 1_{\bar{\rho}} \circ 1_{\bar{\rho}} \times \bar{r} = 1_{\bar{\rho}}, \tag{13}$$

here

$$\bar{r} = \varepsilon(\bar{\rho}, \rho) \circ r. \tag{14}$$

Lemma 1. *Let the triplet $(\bar{\rho}, r, \bar{r})$ defines the conjunction for $\rho \in \mathbf{obj}\,\mathcal{C}$. Then, the mappings*

$$f : s \to 1_{\bar{\rho}} \times s \circ r \times 1_\rho,$$
$$(\rho^2, \rho) \to (\rho, \bar{\rho} \otimes \rho); \tag{15}$$

$$\tilde{f} : s' \to \bar{r}^* \times 1_\rho \circ 1_\rho \times s',$$
$$(\rho, \bar{\rho} \otimes \rho) \to (\rho^2, \rho) \tag{16}$$

are invertible. Here, $1_{\bar{\rho}} \in (\bar{\rho}, \bar{\rho})$, $1_\rho \in (\rho, \rho)$, $s \in (\rho^2, \rho)$, $s' \in (\rho, \bar{\rho} \otimes \rho)$, and ρ^2 refers to $\rho \otimes \rho$.

Proof. To prove the lemma, we will use the scheme presented in [7] for categories with arbitrary morphisms. From the mapping (15), we have the following expression: $1_{\bar{\rho}} \times (\bar{r}^* \times 1_\rho \circ 1_\rho \times s') \circ r \times 1_\rho$. Noting, that $1_{\bar{\rho}} \times 1_\rho = 1_{\bar{\rho}\rho}$, and by applying the alternation rule (8), we obtain the following: $1_{\bar{\rho}} \times (\bar{r}^* \times 1_\rho \circ 1_\rho \times s') \circ r \times 1_\rho = 1_{\bar{\rho}} \times \bar{r}^* \times 1_\rho \circ (1_{\bar{\rho}\rho} \circ r) \times (s' \circ 1_\rho) = 1_{\bar{\rho}} \times \bar{r}^* \times 1_\rho \circ r \times s' = 1_{\bar{\rho}} \times \bar{r}^* \times 1_\rho \circ r \times (1_{\bar{\rho}\rho} \circ s')$. It should be noted that $r \times s' = r \times 1_{\bar{\rho}\rho} \circ s'$. Taking into account the second equation (13) and taking conjugation, we obtain that $1_{\bar{\rho}} \times \bar{r}^* \times 1_\rho \circ r \times (1_{\bar{\rho}\rho} \circ s') = (1_{\bar{\rho}} \times \bar{r}^* \circ r \times 1_{\bar{\rho}}) \times 1_\rho \circ s' = s' \in (\rho, \bar{\rho} \otimes \rho)$. Hence, $f : s \to s'$. Similarly, it is easy to verify that $\tilde{f} = f^{-1} : s' \to s$. The spaces (ρ^2, ρ) and $(\rho, \bar{\rho} \otimes \rho)$ are isomorphic. The lemma is proved.

The invertibility of mappings can be shown similarly:

$$t \to t \times 1_{\bar{\rho}} \circ 1_\rho \times \bar{r}, \quad (\rho^2, \rho) \to (\rho, \rho \otimes \bar{\rho}); \tag{17}$$
$$t' \to 1_\rho \times r^* \circ t' \times 1_\rho, \quad (\rho, \rho \otimes \bar{\rho}) \to (\rho^2, \rho). \tag{18}$$

According to the definition, to construct the conjugate object $\bar{\rho}$ satisfying equations (13), it is necessary to have morphisms $r : \iota \to \bar{\rho} \otimes \rho$ and $\bar{r} : \iota \to \rho \otimes \bar{\rho}$. Below, we will consider a specific category whose objects are endomorphisms of the algebra $\mathcal{A}$, and morphisms are intertwining operators. Let us define the canonical endomorphism

$\rho(a) = \sum_i^d \psi_i a \psi_i^*$, where $a \in \mathcal{A}$, $\psi_i \in \mathcal{O}_d$ (see Subsections 1.1 and 1.4). Then, we define r using the relation $r = \sum_{i=1}^3 \hat{\psi}_i \psi_i$, where

$$\hat{\psi}_1 = \frac{1}{\sqrt{2}} (\psi_2\psi_3 - \psi_3\psi_2),$$
$$\hat{\psi}_2 = \frac{1}{\sqrt{2}} (\psi_3\psi_1 - \psi_1\psi_3),$$
$$\hat{\psi}_3 = \frac{1}{\sqrt{2}} (\psi_1\psi_2 - \psi_2\psi_1) \quad (19)$$

are defined according to (12) when $d = 3$.

Lemma 2. $\bar{r} = \sum_{i=1}^3 \psi_i \hat{\psi}_i$.

Proof. In particular, for the case of the tensor square of Hilbert spaces $\mathcal{H} \otimes \mathcal{H}'$, it follows from (9) that $\vartheta(p_2) = \vartheta(\mathcal{H}, \mathcal{H}') = \sum_{i,j} \psi_i' \psi_j \psi_i'^* \psi_j^*$, $i, j = 1, 2, 3$. Taking into account (14) and using the equivalence[1)] $\varepsilon(\bar{\rho}, \rho) = \vartheta(\hat{\mathcal{H}}, \mathcal{H}) = \sum_{i,j} \psi_i \hat{\psi}_j \psi_i^* \hat{\psi}_j^*$, we get $\bar{r} = \vartheta(\hat{\mathcal{H}}, \mathcal{H}) \circ r = (\sum_{i,j} \psi_i \hat{\psi}_j \psi_i^* \hat{\psi}_j^*) \circ (\sum_{i=1}^3 \psi_i \hat{\psi}_i)$. Using straightforward calculations, we obtain that $\bar{r} = \sum_{i=1}^3 \psi_i \hat{\psi}_i$. The lemma is proved. It is also obvious that $\bar{r}^* = \sum_{i=1}^3 \hat{\psi}_i^* \psi_i^*$.

Let us formulate the following statement.

Statement. *Let $r = \sum_{i=1}^3 \hat{\psi}_i \psi_i$ and $\bar{r} = \sum_{i=1}^3 \psi_i \hat{\psi}_i$. Then, there exists a conjugate object $\bar{\rho}(a) = \sum_{i=1}^3 \hat{\psi}_i a \hat{\psi}_i^*$ such that $r\iota(a) = \bar{\rho} \otimes \rho(a) r$ ($a \in \mathcal{A}$), with conjugation equations satisfied* (13).

Proof. First, let us show that the condition $r\iota(a) = \bar{\rho} \otimes \rho(a) r$ is satisfied (i.e., $r \in (\iota, \bar{\rho} \otimes \rho)$). The left-hand side of this equation is given by $r\iota(a) = ra$. Since $\bar{\rho} \otimes \rho(a) = \sum_{i=1}^3 \hat{\psi}_i \rho(a) \hat{\psi}_i^*$, then, using relations (19) and the expression $r = \sum_{i=1}^3 \hat{\psi}_i \psi_i$, it is easy to see that $\bar{\rho} \otimes \rho(a) = (\sum_{i=1}^3 \hat{\psi}_i \rho(a) \hat{\psi}_i^*) \circ (r = \sum_{i=1}^3 \hat{\psi}_i \psi_i) = (\hat{\psi}_1\psi_1 + \hat{\psi}_2\psi_2 + \hat{\psi}_3\psi_3) a = ra$. Therefore, $r \in (\iota, \bar{\rho} \otimes \rho)$.

Let us now consider the first conjugation equation (13). We transform the expression $\bar{r}^* \times 1_\rho$ using the formula (7). Then, $\bar{r}^* \times 1_\rho = 1_\rho \circ \bar{r}^*$. Considering that $1_\rho = \sum_{i=1}^3 \psi_i \psi_i^* \in (\rho, \rho)$ and $\bar{r}^* = \sum_{i=1}^3 \hat{\psi}_i^* \psi_i^*$, we find the expression for $1_\rho \circ \bar{r}^*$ (due to its cumbersomeness, we do not provide it here). Transforming the expression $1_\rho \times r$, using the same formula (7), we obtain $1_\rho \times r = \rho(r) \circ 1_\rho$. Noting that the composition with 1_ρ on the right does not change $\rho(r)$, we obtain $\rho(r) \circ 1_\rho = \rho(r)$ (Indeed, $\rho(r) \circ 1_\rho = (\sum_{i=1}^3 \psi_i r \psi_i^*) \circ (\sum_{i=1}^3 \psi_i \psi_i^*)) = \rho(r)$). Composing $1_\rho \circ \bar{r}^*$ with $\rho(r) = \sum_{i=1}^3 \psi_i r \psi_i^*$, we get $(1_\rho \circ \bar{r}^*) \circ \rho(r) = 1_\rho$. The first conjugation equation is thus satisfied.

The second conjugation equation is proved similarly. Here we simply note that $1_{\bar{\rho}} = \bar{\rho}(1) \in (\bar{\rho}, \bar{\rho})$.

Corollary. $1_{\bar{\rho}} = A_2^\rho$, where $A_2^\rho \in (\rho^2, \rho^2)$ is a projector onto the antisymmetric subspace of dimension $n = d - 1$ (in our case $d = 3$).

It can be easily shown by following the general formula [7]

$$A_n^\rho = 1/n! \sum_{p \in \mathbb{P}_n} \text{sgn}(p) \varepsilon_\rho(p)$$

when $n = 2$. Then, we obtain that

$$A_2^\rho = 1/2(1_{\rho^2} - \varepsilon(\rho, \rho)). \quad (20)$$

Since $1_{\rho^2} = \rho^2(I) = \rho(\rho(I)) = \sum_{j=1}^3 \psi_j (\sum_{i=1}^3 \psi_i \times \psi_i^*) \psi_j^*$ and $\varepsilon(\rho, \rho) = \vartheta(\mathcal{H}, \mathcal{H}) = \sum_{i,j}^3 \psi_i \psi_j \psi_i^* \psi_j^*$, Direct calculation shows that $1_{\bar{\rho}} = A_2^\rho$.

2.2. Left Inverse for ρ

Given the existence of a conjugate object in the category, using morphisms that satisfy the conjugation equations (13), we can find a specific expression for the left inverse mapping ϕ. By definition (see Subsection 1.2), the left inverse acts according to the rule $\phi^\rho = \{\phi^\rho_{\rho_1, \rho_2} : (\rho \otimes \rho_1, \rho \otimes \rho_2) \longrightarrow (\rho_1, \rho_2)\}$. Let $\phi_{\sigma,\tau}(x) = s^* \times 1_\tau \circ 1_{\bar{\rho}} \times x \circ r \times 1_\sigma$, where $x \in (\rho \otimes \sigma, \rho \otimes \tau)$ and $r, s \in (\iota, \bar{\rho} \otimes \rho)$ [19]. We will show now that $\phi_{\bar{\rho},\bar{\rho}}(x) \in (\bar{\rho}, \bar{\rho})$. By setting $s = r$, where $r \in (\iota, \bar{\rho} \otimes \rho)$, we obtain

$$\phi_{\bar{\rho},\bar{\rho}}(x) = r^* \times 1_{\bar{\rho}} \circ 1_{\bar{\rho}} \times x \circ r \times 1_{\bar{\rho}}, \quad (21)$$

$x \in (\rho \otimes \bar{\rho}, \rho \otimes \bar{\rho})$. Here $1_{\bar{\rho}} = \bar{\rho}(I) = \sum_{i=1}^3 \hat{\psi}_i \hat{\psi}_i^*$. Since by definition (Subsection 2.1) $r = \sum_{i=1}^3 \hat{\psi}_i \psi_i$, $r^* = \sum_{i=1}^3 \psi_i^* \hat{\psi}_i^*$ and according to, $\bar{\rho}(a) = \sum_{i=1}^3 \hat{\psi}_i a \hat{\psi}_i^*$, it is easy to show that

(a) $r^* \times 1_{\bar{\rho}} = \iota(1_{\bar{\rho}}) \circ r^* = 1_{\bar{\rho}} \circ r^* = \sum_{i,j} \hat{\psi}_i \hat{\psi}_i^* \times \psi_j^* \hat{\psi}_j^*$ (where we used the property (7)).

(b) $1_{\bar{\rho}} \times x = \bar{\rho}(x) = \sum_k \hat{\psi}_k x \hat{\psi}_k^*$.

(c) Once again, using (7), we obtain

$r \times 1_{\bar{\rho}} = r \circ \iota(1_{\bar{\rho}}) = r \circ 1_{\bar{\rho}} = \sum_{m,n} \hat{\psi}_m \psi_m \hat{\psi}_n \hat{\psi}_n^*$.

Substituting (a)–(c) into (21), we obtain

$$\phi_{\bar{\rho},\bar{\rho}}(x) = 1_{\bar{\rho}} \circ \sum_j \psi_j^* x \psi_j \circ 1_{\bar{\rho}}$$
$$= \sum_j \psi_j^* x \psi_j \in (\bar{\rho}, \bar{\rho}). \quad (22)$$

[1)]Due to the mentioned isomorphism $\mathbf{hilb}_G \simeq (\mathcal{C}, \varepsilon)$, see Subsection 2.4.

Since $x \in (\rho \otimes \bar{\rho}, \rho \otimes \bar{\rho})$, this can be equivalently represented as $x = \rho(t)$, where $t \in (\bar{\rho}, \bar{\rho})$. Therefore, $\sum_j \psi_j^* x \psi_j = \sum_j \sum_i \psi_j^* \psi_i t \psi_i^* \psi_j = \sum_j \sum_i \delta_{ji} t \delta_{ij} = dt$ and expression (22) is the left inverse of ρ. However, the requirement $\phi(I) = I$ is not satisfied here, that is why we introduce the normalized left inverse by $\tilde{\phi}(x) = \frac{1}{d}\phi(x)$. Then, we obtain that

$$\tilde{\phi}_{\bar{\rho},\bar{\rho}}(x) = \frac{1}{d} \sum_j \psi_j^* x \psi_j. \tag{23}$$

Let us consider the action of the left inverse mapping on the projection operator $A^{\rho}_{d=3}$ onto the antisymmetric subspace of the space (ρ^3, ρ^3). Using (10), we have $A^{\rho}_3 = SS^*$, where $S = \frac{1}{\sqrt{3}} \sum_i \psi_i \hat{\psi}_i$, $S^* = \frac{1}{\sqrt{3}} \sum_j \hat{\psi}_j^* \psi_j^*$. Then, $A^{\rho}_3 = SS^* = \frac{1}{3} \sum_{i,j} \psi_i \times \hat{\psi}_i \hat{\psi}_j^* \psi_j^*$ and using (22), we have $\phi(A^{\rho}_3) = \frac{1}{3} \sum_{i,j,k} \psi_k^* \psi_i \hat{\psi}_i \hat{\psi}_j^* \psi_j^* \psi_k$. Using the Cuntz relations (1) and (2), we finally obtain $\phi(A^{\rho}_3) = \frac{1}{3} \sum_i \hat{\psi}_i \hat{\psi}_i^* = \frac{1}{3} 1_{\bar{\rho}} = \frac{1}{3} A^{\rho}_2$.

2.3. Sector Statistics

The left inverse allows for describing the statistics of the sector ρ using the statistical parameter $\lambda_\rho = \phi(\varepsilon(\rho, \rho))$. For an irreducible ρ, we have $\phi(\varepsilon(\rho, \rho)) = \lambda_\rho \mathbb{I}$, where $\lambda_\rho \in \{0\} \cup \{\pm d^{-1} : d \in \mathbb{N}\}$, and $\lambda_\rho = \frac{1}{d}$ corresponds to a parabose statistic of order d with a Young tableau, having column length $\leq d$, $\lambda_\rho = -\frac{1}{d}$ corresponds to parafermi statistics of order d with a Young tableau with a row length $\leq d$. The case $\lambda_\rho = 0$ describes infinite statistics, which is not observed for real particles. [2)] The natural number d is called the statistical dimension of the superselection sector, which coincides with the concept of dimension $\dim(\rho)$ of the object ρ of the symmetric tensor category.

In the case $d = 3$ for one particle, we have

$$\phi(\varepsilon(\rho, \rho)) = \phi(\vartheta) = \frac{1}{3} \sum_{k,i,j} \psi_k^* \psi_i \psi_j \psi_i^* \psi_j^* \psi_k$$
$$= \frac{1}{3} \sum_i \psi_i \psi_i^* = \frac{1}{3} I,$$

which defines a sector with parabose statistics of statistical dimension 3.

The definition of the statistics of multiparticle sectors also requires the decomposition into a direct sum of tensor products of endomorphisms (including their conjugates) and the determination of $3j$-symbols. For example, for $d = 3$, we have the decomposition $\rho \otimes \rho = \rho_0 \oplus \rho_1$, where ρ_0 has dimension $d = 1$ and ρ_1 has dimension $d = 8$. This means that the "collision" of two particles with $d = 3$ results in a Bose particle and a particle with parastatistics of order 8. However, a detailed study of multiparticle sectors and the classification of their statistics is only necessary when considering specific issues related to quantum information transfer and quantum cryptography. Therefore, we will explore them in a separate publication dedicated to these topics.

2.4. Algebra of Observables

The abstract tensor C^*-category of canonical endomorphisms $\mathcal{C}_\rho$ that we have studied is closed under direct sums and subobjects, and it possesses conjugation and symmetry. In Subsection 1.4, we mentioned that such a category allows for constructing the crossed product $\mathcal{A} \times \mathcal{C} = \mathcal{F} \supset \mathcal{A}$ using the construction developed in [11]. This algebra contains $\mathcal{A}$ as a subalgebra that remains pointwise fixed under the action of the compact group G, where G is a closed subgroup of the group $aut(\mathcal{A} \times \mathcal{C})$. In physical terms, the compact group G corresponds to the global gauge group of transformations [20].

Without going into details regarding norm and continuity, let us consider the conditional expectation, which is a positive linear mapping (an idempotent) onto the pointwise fixed subalgebra $\mathcal{A}$ with under the action of the compact group G by $m(\mathcal{F}) = \int \alpha_g(B) d\mu(g)$ [16]. Here, $\alpha : g \to \alpha_g$, $g \in G$, $\alpha_g \in aut(\mathcal{F})$, $B \in \mathcal{F}$, and $d\mu(g)$ is a normalized Haar measure.

Here, α_g can be defined as $\alpha_g \psi = u(g)\psi$, where $u(g)$ is a unitary unimodular matrix, which, in the case of the $SU(2)$ group takes the form:

$$u(g) = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \tag{24}$$

whose matrix elements satisfy the condition $|\alpha|^2 + |\beta|^2 = 1$ and $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$. Since $\alpha_g(\mathcal{A}) = \mathcal{A}$, in the case of the $SU(d)$ group, we have $\alpha_g \rho(A) = \rho \alpha_g(A) = \rho(A)$, where $A \in \mathcal{A}$, $\rho(A) = \sum_{i=1}^d \psi_i A \psi_i^*$. If, for example, $\mathcal{F} = \mathcal{O}_2$, then, using (24) and parameterizing the matrix elements as

$$\alpha = \cos\frac{\theta}{2} \exp\left(i\frac{\varphi_1 + \varphi_2}{2}\right);$$
$$\beta = i \sin\frac{\theta}{2} \exp\left(-i\frac{\varphi_2 - \varphi_1}{2}\right);$$

$$\bar{\alpha} = \cos\frac{\theta}{2} \exp\left(-i\frac{\varphi_1 + \varphi_2}{2}\right);$$

[2)]This case describes anions.

$$\bar{\beta} = -i \sin \frac{\theta}{2} \exp \left(i \frac{\varphi_2 - \varphi_1}{2} \right), \quad (25)$$

where $\varphi_1, \varphi_2, \theta$ are Euler angles ($0 \leq \varphi_1 \leq 2\pi$, $0 \leq \varphi_2 \leq 2\pi$, $0 \leq \theta \leq \pi$), it can be easily shown that $\mathcal{O}_{SU(2)} = m(\mathcal{O}_2)$.

In the case of the conjugate endomorphism $\bar{\rho}$, we are dealing with the conjugate representation, and the isometric operators ψ are replaced by the equalities (12). Using similar calculations, one can obtain a new algebra $\mathcal{O}_{\widehat{SU(3)}}$, which is invariant under the conjugate representation of the group $SU(3)$. This allows for generalizing the algebra of observables in the presence of a conjugate object.

2.5. Quantum Three-Level System

Following the scheme developed in [14] for the case of a two-level system with isospin $T = 1/2$, let us consider a three-level quantum system (qutrit) whose state space is a three-dimensional Hilbert space $\mathcal{H}$ formed by the linear span $\mathcal{H} = \mathrm{Lin}\{\psi_i\}_{i=1}^3$ of the multiplet ψ_1, ψ_2, ψ_3 (see Subsection 1.1). This multiplet serves as an orthonormal basis in the space $\mathcal{H}$, where the fundamental representation π of the $SU(3)$ group is realized.

The state space of two qutrits decomposes into a direct sum of two coherent subspaces $\mathcal{H} \otimes \mathcal{H} = \mathcal{H}_6 \oplus \bar{\mathcal{H}}_3$, each of which is acted upon by irreducible representations π_6 and $\bar{\pi}_3$ of the $SU(3)$ group, where $\pi \otimes \pi = \pi_6 \oplus \bar{\pi}_3$. The basis of the space $\mathcal{H}_6$ consists of symmetric tensors formed from the basis elements ψ_1, ψ_2, ψ_3 satisfying relations (1), (2):

$$\left. \begin{aligned} \psi_{11} &= \psi_1 \psi_1 \\ \psi_{12} &= \frac{1}{\sqrt{2}} (\psi_1 \psi_2 + \psi_2 \psi_1) \\ \psi_{13} &= \frac{1}{\sqrt{2}} (\psi_1 \psi_3 + \psi_3 \psi_1) \\ \psi_{22} &= \psi_2 \psi_2 \\ \psi_{23} &= \frac{1}{\sqrt{2}} (\psi_2 \psi_3 + \psi_3 \psi_2) \\ \psi_{33} &= \psi_3 \psi_3 \end{aligned} \right\}. \quad (*)$$

The basis of the space $\bar{\mathcal{H}}_3$, in which the conjugate representation $\bar{\pi}_3$ of the $SU(3)$ group acts, is determined by the expression:

$$\left. \begin{aligned} \hat{\psi}_{23} &= \frac{1}{\sqrt{2}} (\psi_2 \psi_3 - \psi_3 \psi_2) \\ \hat{\psi}_{31} &= \frac{1}{\sqrt{2}} (\psi_3 \psi_1 - \psi_1 \psi_3) \\ \hat{\psi}_{12} &= \frac{1}{\sqrt{2}} (\psi_1 \psi_2 - \psi_2 \psi_1) \end{aligned} \right\}. \quad (**)$$

Thus, the 9-dimensional state space is divided into two superselection sectors, one of which is the conjugate sector. The projectors onto the basis states of the $\mathcal{H}_6$ space are determined by the expressions:

$$\Pi_{11} = \psi_{11}\psi_{11}^*, \quad \Pi_{12} = \psi_{12}\psi_{12}^*, \quad \Pi_{13} = \psi_{13}\psi_{13}^*,$$
$$\Pi_{22} = \psi_{22}\psi_{22}^*, \quad \Pi_{23} = \psi_{23}\psi_{23}^*,$$
$$\Pi_{33} = \psi_{33}\psi_{33}^*. \quad (26)$$

Similarly, for the projectors onto the basis states of the $\bar{\mathcal{H}}_3$ space, we obtain the expressions:

$$\hat{\Pi}_{23} = \hat{\psi}_{23}\hat{\psi}_{23}^*, \quad \hat{\Pi}_{31} = \hat{\psi}_{31}\hat{\psi}_{31}^*,$$
$$\hat{\Pi}_{12} = \hat{\psi}_{12}\hat{\psi}_{12}^*. \quad (27)$$

Without going into details, using the framework developed in [14], it can be shown, for example, that the state prepared by Alice in the symmetric state ψ_{11} in her coordinate system will be obtained by Bob after the averaging procedure as a (mixed) state:

$$\tilde{\Pi}_{11} = \frac{1}{6}(\Pi_{11} + \Pi_{12} + \Pi_{13} + \Pi_{23} + \Pi_{22} + \Pi_{33}). \quad (28)$$

In obtaining this expression, we used the averaging procedure over the $SU(3)$ group with the Haar measure

$$d\mu(g) = \frac{4\sqrt{3}}{\pi^5} \times \sin 2\alpha_2 \cos \alpha_4 \sin^3 \alpha_4 \sin 2\alpha_6 d\alpha_1 d\alpha_2 \times \ldots \times d\alpha_8,$$

as described in the work [21], where

$$0 \leq \alpha_1 \alpha_3, \alpha_5, \alpha_7 \leq \pi; \quad 0 \leq \alpha_1, \alpha_2, \alpha_4, \alpha_6 \leq \pi/2;$$
$$0 \leq \alpha_8 \leq \pi/\sqrt{3}$$

are generalized Euler angles. It can be easily verified that $[\bar{\Pi}_{11}, G] = 0$, which indicates that $\tilde{\Pi}_{11}$ belongs to the algebra of observables.

We plan to conduct a more detailed analysis of the superselection structure of the algebra $\mathcal{O}_{SU(3)}$ in the presence of conjugate superselection sectors and its role in quantum information transfer in our future publication.

CONCLUSIONS

In this work, we have investigated the subcategory of the category of endomorphisms with dimension $d = 3$ in the presence of a conjugate object, which, due to the isomorphism between the group G representation category and the abstract symmetric tensor C^*-category, corresponds to the conjugate representation. The presence of such conjugate objects associated with non-Abelian charges favours a richer superselection structure in the theory, since it introduces conjugate sectors alongside the ordinary ones. We expect that these conjugate non-Abelian sectors will play a significant role in quantum information transfer as well as in the formulation of quantum cryptographic protocols.

FUNDING

The work was supported by the Strategic Academic Leadership Program of Kazan (Volga Region) Federal University (Priority-2030).

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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