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On a Semigroup C^*-Algebra for a Semidirect Product

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Abstract—The paper deals with the reduced semigroup C^*-algebra for the semidirect product of a semigroup S by a group G. We represent this C^*-algebra as a reduced crossed product of the reduced semigroup C^*-algebra for S by G. The purpose of the paper is to demonstrate that the crossed product C^*-algebras and the semidirect products of semigroups are closely related. We prove that the action of the group G on the semigroup S can be extended from S to the reduced semigroup C^*-algebra $C_r^*(S)$. We show that the reduced semigroup C^*-algebra for a semidirect product $S \rtimes G$ is isomorphic to the reduced crossed product C^*-algebra $C_r^*(S) \rtimes_r G$. We apply this result to the study of the structure of the reduced semigroup C^*-algebra for the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}^\times$ of the additive group $\mathbb{Z}$ of all integers and the multiplicative semigroup $\mathbb{Z}^\times$ of integers without zero.

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1. INTRODUCTION

In this paper we study the reduced semigroup C^*-algebra for a semidirect product of a semigroup S by a group G. The main purpose of our work is to represent this C^*-algebra as a reduced crossed product of the reduced semigroup C^*-algebra $C_r^*(S)$ by G.

The reduced semigroup C^*-algebras are very natural objects. They are generated by the left regular representations of semigroups with the cancellation property. The start in studying these algebras was made by Coburn [1, 2] who considered the reduced semigroup C^*-algebra for the additive semigroup of the non-negative integers. Douglas [3] investigated the case of subsemigroups in the additive group of the real numbers. Murphy [4, 5] generalized the results from [1—3] to the case of the reduced semigroup C^*-algebras for the positive cones in ordered groups. For extensive literature and history of the study of semigroup C^*-algebras, the reader is referred, for example, to [6] and the references therein.

The subject of the crossed products C^*-algebras is a well-developed branch of the theory of C^*-algebras. On the one hand, the crossed products provide interesting examples of C^*-algebras. On the other hand, the problem of representing a given C^*-algebra as a crossed product C^*-algebra attract a great deal of attention because it has important applications to a variety of questions in the theory of C^*-algebras. A systematic exposition of the crossed products is contained in the monograph [7].

There are two types of the crossed products of a C^*-algebra $\mathcal{A}$ by a locally compact group G. Namely, these are the full and the reduced crossed products. The full crossed product $\mathcal{A} \rtimes_\alpha G$ should be thought as a twisted maximal tensor product of $\mathcal{A}$ with the full group C^*-algebra $C^*(G)$ of the group G. The reduced crossed product $\mathcal{A} \rtimes_{\alpha,r} G$ should be regarded as a twisted minimal (or spacial) tensor product of $\mathcal{A}$ by the reduced group C^*-algebra $C_r^*(G)$.

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Our research was motivated by the relationship between the crossed products of algebras by groups and the semidirect products of groups. Suppose that H and G are locally compact groups and $\beta : G \longrightarrow \mathrm{Aut}(H)$ is a homomorphism such that an action $(g, h) \mapsto \beta_g(h)$ is continuous from the direct product $G \times H$ to H. Then, the semidirect product $H \rtimes_\beta G$ is the locally compact group. The action β of the group G can be extended from the group H to the C^*-algebra $C^*(H)$ (or $C^*_r(H)$). Denote this action by α. Then, there are natural isomorphisms ([8], II.10.3.15)

$$C^*(H \rtimes_\beta G) \cong C^*(H) \rtimes_\alpha G \quad \text{and} \quad C^*_r(H \rtimes_\beta G) \cong C^*_r(H) \rtimes_{\alpha,r} G.$$

In this paper, we will obtain an analogue of the second isomorphism for the reduced semigroup C^*-algebra of a discrete semigroup. Namely, let S be a discrete left cancellative semigroup, G be a discrete group and $\beta : G \longrightarrow \mathrm{Aut}(S)$ be a group homomorphism. Then, the semidirect product $S \rtimes_\beta G$ is the left cancellative semigroup. In Section 2 we will see that there exists an isomorphism

$$C^*_r(S \rtimes_\beta G) \cong C^*_r(S) \rtimes_{\alpha,r} G,$$

where $\alpha : G \longrightarrow \mathrm{Aut}(C^*_r(S))$ is the group homomorphism induced by the homomorphism β. In Section 3, the above result will be applied to the reduced semigroup C^*-algebra $C^*_r(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ which was studied in [9–11].

2. PRELIMINARIES

We begin by recalling the definition of the reduced semigroup C^*-algebra for a semigroup.

Let S be a discrete left cancellative semigroup. As usual, the symbol $l^2(S)$ stands for the Hilbert space of all square summable complex-valued functions on S. For every $a \in S$, we denote by e_a the function in $l^2(S)$ which is defined as follows: $e_a(b) = 1$, if $a = b$, and $e_a(b) = 0$, if $a \neq b$, where $b \in S$. Then, the set of functions $\{e_a \mid a \in S\}$ is an orthonormal basis in the Hilbert space $l^2(S)$.

In the C^*-algebra of all bounded linear operators $B(l^2(S))$ on the Hilbert space $l^2(S)$, we define the C^*-subalgebra $C^*_r(S)$ generated by the set of isometries $\{T_a \mid a \in S\}$, where $T_a(e_b) = e_{ab}$ for $a, b \in S$. It is called *the reduced semigroup C^*-algebra*. The identity element in this algebra is denoted by I.

Now we recall the necessary notions concerning the crossed products of C^*-algebras by locally compact groups [7, 8].

Let $\mathcal{A}$ be a C^*-algebra, G be a locally compact group and $\alpha : G \longrightarrow \mathrm{Aut}(\mathcal{A})$ be a continuous homomorphism of groups. The triple $(\mathcal{A}, G, \alpha)$ is called *a dynamical system*.

A covariant representation of the dynamical system $(\mathcal{A}, G, \alpha)$ is a pair (π, u) consisting of a nondegenerate representation $\pi : \mathcal{A} \longrightarrow B(H)$ and a unitary representation $u : G \longrightarrow B(H)$ for a Hilbert space H such that

$$\pi(\alpha_g(a)) = u(g)\pi(a)u(g)^*$$

for all $a \in \mathcal{A}$ and $g \in G$ [7, Def. 2.10].

Let $C_c(G, \mathcal{A})$ be the space of finitely supported functions $f : G \longrightarrow \mathcal{A}$. This space becomes the $*$-algebra with a convolution and an involution twisted with using the homomorphism α [7, p. 48]. The Banach algebra $L^1(G, \mathcal{A})$ is the completion of $C_c(G, \mathcal{A})$ with respect to the L^1-norm. If G is a discrete group, then $C_c(G, \mathcal{A}) = \mathcal{A}G$ is an $*$-algebra of finite linear combinations of elements of the group G with coefficients from $\mathcal{A}$.

If (π, u) is a covariant representation of $(\mathcal{A}, G, \alpha)$ on H, then there exists an associated $*$-representation $\pi \rtimes u : C_c(G, \mathcal{A}) \longrightarrow B(H)$ such that $||(\pi \rtimes u)f|| \leq ||f||_1$, where $f \in C_c(G, \mathcal{A})$ [7, Prop. 2.23]. The completion of $C_c(G, \mathcal{A})$ with respect to the universal norm

$$||f|| := \sup\{||(\pi \rtimes u)f|| : (\pi, u) \text{ is a covariant representation of } (\mathcal{A}, G, \alpha)\}$$

is called *the (full) crossed product of $\mathcal{A}$ by G* and denoted by $\mathcal{A} \rtimes_\alpha G$ [7, Lem. 2.27].

The term "crossed product" will always mean "full crossed product".

It is worth noting that for every dynamical system $(\mathcal{A}, G, \alpha)$ there exists a crossed product. Moreover, it is unique up to an isomorphism. For proving these facts we refer the reader to [12].

Next let us define the reduced crossed product C^*-algebra [8].

Let $\pi : \mathcal{A} \longrightarrow B(H)$ be a faithful representation. Define representations $\pi_\alpha : \mathcal{A} \longrightarrow B(L^2(G, H))$ and $\lambda : G \longrightarrow B(L^2(G, H))$ as follows

$$(\pi_\alpha(a)\chi)(h) = \pi(\alpha_{h^{-1}}(a))(\chi(h)), \tag{1}$$

$$(\lambda(g)\chi)(h) = \chi(g^{-1}h), \tag{2}$$

where $a \in \mathcal{A}$, $g, h \in G$, $\chi \in L^2(G, H)$. Then, the pair (π_α, λ) is a covariant representation of the dynamical system $(\mathcal{A}, G, \alpha)$ on the Hilbert space $L^2(G, H)$. The *reduced norm* on $C_c(G, \mathcal{A}) \subset L^1(G, \mathcal{A})$ is given by

$$||f||_r := ||(\pi_\alpha \rtimes \lambda)f||,$$

where $f \in C_c(G, \mathcal{A})$. The completion of $C_c(G, \mathcal{A})$ with respect to $|| \cdot ||_r$ is called *the reduced crossed product of $\mathcal{A}$ by G* and denoted by $\mathcal{A} \rtimes_{\alpha,r} G$.

If G is a discrete group, then $L^2(G, H) = l^2(G, H) \cong H \otimes l^2(G)$. The formulas (1) and (2) can be rewritten as follows

$$\pi_\alpha(a)(\xi \otimes e_h) = \pi(\alpha_{h^{-1}}(a))\xi \otimes e_h, \quad \lambda(g)(\xi \otimes e_h) = \xi \otimes e_{gh},$$

where $a \in \mathcal{A}$, $g, h \in G$, $\xi \in H$. The set of functions $\{e_h \mid h \in G\}$ is an orthonormal basis in the Hilbert space $l^2(G)$. If G is a discrete group and $\mathcal{A}$ is a unital C^*-algebra, then one can say that the reduced crossed product $\mathcal{A} \rtimes_{\alpha,r} G$ is generated by the set $\{\pi_\alpha(a) | a \in \mathcal{A}\} \cup \{\lambda(g) | g \in G\}$.

If G is amenable, then the reduced norm coincides with the universal norm on $C_c(G, \mathcal{A})$ and we have $\mathcal{A} \rtimes_{\alpha,r} G = \mathcal{A} \rtimes_\alpha G$ [7, Th. 7.13].

3. THE SEMIGROUP C^*-ALGEBRA $C_r^*(S \rtimes_\beta G)$

Let S be a discrete left cancellative semigroup, and G be a discrete group. Let $\beta : G \longrightarrow \mathrm{Aut}(S)$ be a group homomorphism. Then, the semidirect product $S \rtimes_\beta G$ is the semigroup with the underlying set $S \times G$ and the semigroup operation given by

$$(a, g)(b, h) := (a\beta_g(b), gh).$$

It is easy to see that the semigroup $S \rtimes_\beta G$ has the left cancellation property. Here, the object of our study is the reduced semigroup C^*-algebra $C_r^*(S \rtimes_\beta G)$. Let us fix an arbitrary element $s \in S$ and introduce the notation $U_{g,s} := T^*_{(s,e)} T_{(s,g)}$ and $V_a := T_{(a,e)}$, where $g \in G$, $a \in S$ and e is the unit of the group G. We show that the action of the operator $U_{g,s}$ on the space $l^2(S \rtimes_\beta G)$ does not depend on the choice of the element s. To do this we find out how this operator acts on basis vectors. We have

$$U_{g,s} e_{(a,h)} = T^*_{(s,e)} T_{(s,g)} e_{(a,h)} = T^*_{(s,e)} e_{(s\beta_g(a),gh)} = T^*_{(s,e)} T_{(s,e)} e_{(\beta_g(a),gh)} = e_{(\beta_g(a),gh)}, \tag{3}$$

where $g, h \in G$, $a \in S$. Thus, the action of the operator $U_{g,s}$ on basis vectors has nothing to do with the element s. So the operator $U_{g,s}$ is denoted by U_g.

Lemma 1. *The following properties are fulfilled:*

1) *The operator U_g is unitary for every $g \in G$;*

2) *The C^*-algebra $C_r^*(S \rtimes_\beta G)$ is generated by the unitary operators U_g, $g \in G$, and the isometries V_a, $a \in S$.*

Proof. 1) First we calculate the values of the operator U_g^* at the basis vectors

$$U_g^* e_{(a,h)} = T^*_{(s,g)} T_{(s,e)} e_{(a,h)} = T^*_{(s,g)} e_{(sa,h)} = T^*_{(s,g)} T_{(s,g)} e_{(\beta_{g^{-1}}(a),g^{-1}h)} = e_{(\beta_{g^{-1}}(a),g^{-1}h)}, \tag{4}$$

where $g, h \in G$, $a \in S$. Next, using (3) and (4), we get

$$U_g U_g^* e_{(a,h)} = U_g e_{(\beta_{g^{-1}}(a),g^{-1}h)} = e_{(a,h)}, \quad U_g^* U_g e_{(a,h)} = U_g^* e_{(\beta_g(a),gh)} = e_{(a,h)}$$

for all $g, h \in G$, $a \in S$. Hence, we have $U_g^* U_g = U_g U_g^* = I$.

2) The statement follows from the following representation

$$T_{(a,g)} = T^*_{(s,e)}T_{(s,e)}T_{(a,g)} = T^*_{(s,e)}T_{(sa,g)} = T^*_{(s,e)}T_{(s,g)}T_{(\beta_{g^{-1}}(a),e)} = U_g V_{\beta_{g^{-1}}(a)},$$

where $a \in S$, $g \in G$. We note that using the actions of operators $T_{(a,g)}$, V_a and U_g on the basis vectors, it is easy to show one more equality $T_{(a,g)} = V_a U_g$. □

Further we consider the C^*-algebra $C^*_r(S)$. In the next lemma it will be shown that any automorphism of the semigroup S induces an automorphism of the semigroup C^*-algebra $C^*_r(S)$.

Lemma 2. *Let $\gamma : S \longrightarrow S$ be an automorphism of the semigroup S. Then, there exists a unique automorphism $\overline{\gamma} : C^*_r(S) \longrightarrow C^*_r(S)$ such that $\overline{\gamma}(T_a) = T_{\gamma(a)}$ whenever $a \in S$.*

Proof. Consider the unitary operator $U : l^2(S) \longrightarrow l^2(S) : e_b \mapsto e_{\gamma(b)}$, $b \in S$, and the isometric $*$-homomorphism $\tilde{\gamma} : C^*_r(S) \longrightarrow B(l^2(S)) : A \longmapsto UAU^*$, $A \in C^*_r(S)$. It is easy to verify that the equality $T_{\gamma(a)} = UT_aU^*$ holds for each $a \in S$. Hence, we have $\tilde{\gamma}(T_a) = T_{\gamma(a)}$ whenever $a \in S$. Since γ is an automorphism of S, the image of $\tilde{\gamma}$ contains the dense $*$-subalgebra of the C^*-algebra $C^*_r(S)$. Therefore the image of $\tilde{\gamma}$ is dense in $C^*_r(S)$. Denote by $\overline{\gamma}$ the corestriction of $\tilde{\gamma}$ to $C^*_r(S)$. Of course, $\overline{\gamma}$ is an automorphism of the C^*-algebra $C^*_r(S)$. The uniqueness of the required automorphism is obvious. □

Thus, if $\beta : G \longrightarrow \mathrm{Aut}(S)$ is a group homomorphism, then we have the group homomorphism $\alpha : G \longrightarrow \mathrm{Aut}(C^*_r(S))$ such that $\alpha_g(T_a) = T_{\beta_g(a)}$ for all $g \in G$, $a \in S$. So we have the dynamical system $(C^*_r(S), G, \alpha)$.

Next, let us construct the reduced crossed product $C^*_r(S) \rtimes_{\alpha,r} G$. Firstly, using the inclusion $C^*_r(S) \subset B(l^2(S))$, we define a representation $\pi : C^*_r(S) \longrightarrow B(l^2(S) \otimes l^2(G))$ on generators of the C^*-algebra $C^*_r(S)$ as follows:

$$\pi(T_a)(e_b \otimes e_g) = \alpha_{g^{-1}}(T_a)e_b \otimes e_g = e_{\beta_{g^{-1}}(a)b} \otimes e_g, \tag{5}$$

where $a, b \in S$, $g \in G$. Secondly, we define a regular representation $\lambda : G \longrightarrow B(l^2(S) \otimes l^2(G))$ by

$$\lambda(g)(e_b \otimes e_h) = e_b \otimes e_{gh}, \tag{6}$$

where $b \in S$, $g, h \in G$. Then, the pair (π, λ) is a covariant representation of the dynamical system $(C^*_r(S), G, \alpha)$. Since the C^*-algebra $C^*_r(S)$ is unital and the group G is discrete, the C^*-algebra $C^*_r(S) \rtimes_{\alpha,r} G$ is generated by the set $\{\pi(A) | A \in C^*_r(S)\} \cup \{\lambda(g) | g \in G\}$. Therefore, because the C^*-algebra $C^*_r(S)$ is generated by the set of operators $\{T_a | a \in S\}$, one can see that the C^*-algebra $C^*_r(S) \rtimes_{\alpha,r} G$ is generated by the set $\{\pi(T_a) | a \in S\} \cup \{\lambda(g) | g \in G\}$.

Theorem 1. *Let S be a discrete left cancellative semigroup and G be a discrete group. Let $\beta : G \longrightarrow Aut(S)$ and $\alpha : G \longrightarrow Aut(C^*_r(S))$ be group homomorphisms such that $\alpha_g(T_a) = T_{\beta_g(a)}$ for all $g \in G$, $s \in S$. Then, there exists an isomorphism of C^*-algebras*

$$C^*_r(S \rtimes_\beta G) \cong C^*_r(S) \rtimes_{\alpha,r} G.$$

Proof. Let us consider the operator $U : l^2(S) \otimes l^2(G) \longrightarrow l^2(S \rtimes_\beta G)$ defined by the formula

$$U(e_a \otimes e_g) = e_{(\beta_g(a),g)}, \tag{7}$$

where $a \in S$, $g \in G$. Obviously, U is a unitary operator.

Furthermore, we claim that the following diagrams are commutative:

$$\begin{tikzcd} l^2(S) \otimes l^2(G) \arrow[r, "\lambda(g)"] \arrow[d, "U"'] & l^2(S) \otimes l^2(G) \arrow[d, "U"] \\ l^2(S \rtimes_\beta G) \arrow[r, "U_g"] & l^2(S \rtimes_\beta G) \end{tikzcd}$$

for every $g \in G$, and

$$\begin{tikzcd} l^2(S) \otimes l^2(G) \arrow[r, "\pi(T_a)"] \arrow[d, "U"'] & l^2(S) \otimes l^2(G) \arrow[d, "U"] \\ l^2(S \rtimes_\beta G) \arrow[r, "V_a"] & l^2(S \rtimes_\beta G) \end{tikzcd}$$

for every $a \in S$.

Indeed, using (3) and (7), we get

$$U_g U(e_b \otimes e_h) = U_g e_{(\beta_h(b),h)} = e_{(\beta_g(\beta_h(b)),gh)} = e_{(\beta_{gh}(b),gh)},$$

where $b \in S$, $g, h \in G$. On the other hand, by (6) and (7), we have

$$U\lambda(g)(e_b \otimes e_h) = U(e_b \otimes e_{gh}) = e_{(\beta_{gh}(b),gh)}.$$

Thus, the commutativity of the first diagram is shown.

Consider the second diagram. On the one hand, we have

$$V_a U(e_b \otimes e_h) = T_{(a,e)} e_{(\beta_h(b),h)} = e_{(a\beta_h(b),h)},$$

where $a, b \in S$, $h \in G$. On the other hand, using (5), we get

$$U\pi(T_a)(e_b \otimes e_h) = U(e_{\beta_{h^{-1}}(a)b} \otimes e_h) = e_{(\beta_h(\beta_{h^{-1}}(a)b),h)} = e_{(a\beta_h(b),h)}.$$

The commutativity of the diagram is proved, as claimed. Therefore, the equalities

$$\lambda(g) = U^* U_g U, \quad \pi(T_a) = U^* V_a U \tag{8}$$

are true for all $g \in G$ and $a \in S$ respectively.

Further we define the isometric $*$-homomorphism

$$\phi : C_r^*(S \rtimes_\beta G) \longrightarrow B(l^2(S) \otimes l^2(G)) : A \longmapsto U^* A U,$$

where $A \in C_r^*(S \rtimes_\beta G)$. By (8), we have

$$\phi(U_g) = \lambda(g), \quad \phi(V_a) = \pi(T_a)$$

whenever $g \in G$ and $a \in S$.

The image of ϕ is dense in the C^*-algebra $C_r^*(S) \rtimes_{\alpha,r} G$. It follows from the fact that the C^*-algebra $C_r^*(S) \rtimes_{\alpha,r} G$ is generated by the set $\{\pi(T_a) | a \in S\} \cup \{\lambda(g) | g \in G\}$. Thus, the homomorphism ϕ realizes the required isomorphism of C^*-algebras $C_r^*(S \rtimes_\beta G)$ and $C_r^*(S) \rtimes_{\alpha,r} G$. $\square$

4. THE SEMIGROUP C^*-ALGEBRA $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$

In this section we apply Theorem 1 to the study of the structure of the reduced semigroup C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$.

As usual, we denote by $\mathbb{Z}$ the additive group of all integers. Let $\mathbb{Z}^\times$ be the multiplicative semigroup $\mathbb{Z} \setminus \{0\}$ and let $\varphi : \mathbb{Z}^\times \longrightarrow \mathrm{End}(\mathbb{Z})$ be the semigroup homomorphism from $\mathbb{Z}^\times$ into the semigroup of endomorphisms of the group $\mathbb{Z}$ given by

$$\varphi_m(n) := \begin{cases} n, & \text{if } m > 0; \\ -n, & \text{if } m < 0, \end{cases}$$

where $m \in \mathbb{Z}^\times$, $n \in \mathbb{Z}$. We consider the semidirect product of $\mathbb{Z}$ and $\mathbb{Z}^\times$ with respect to φ which is denoted by $\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times$. It is a semigroup with respect to the multiplication defined by

$$(m, n)(k, l) = (m + \varphi_n(k), nl),$$

where $m, k \in Z$, $n, l \in Z^\times$. It is straightforward to verify that $\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times$ is a semigroup with the cancellation property.

The reduced semigroup C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ of the semidirect product $\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times$ is studied in [9–11].

Let $\mathbb{Z} \times \mathbb{N}$ be the Cartesian product of the additive group of all integers and the multiplicative semigroup of natural numbers. It is a semigroup under the multiplication $(m, n)(k, l) = (m + k, nl)$, where $m, k \in \mathbb{Z}, n, l \in \mathbb{N}$.

Let $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ be the cyclic group of order two. Let us define the homomorphism of groups $\alpha : \mathbb{Z}_2 \longrightarrow \mathrm{Aut}(C_r^*(\mathbb{Z} \times \mathbb{N}))$. We put $\alpha_0 = id$, and α_1 is well-defined by the action on the generating elements of the C^*-algebra $C_r^*(\mathbb{Z} \times \mathbb{N})$ as follows: $\alpha_1(T_{(m,n)}) = T_{(-m,n)}$ for all $m \in \mathbb{Z}$, $n \in \mathbb{N}$.

The semigroup C^*-algebras $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$, $C_r^*(\mathbb{Z} \times \mathbb{N})$ and the dynamical system $(C_r^*(\mathbb{Z} \times \mathbb{N}), \mathbb{Z}_2, \alpha)$ were considered in [10]. As a consequence of Theorem 1, we obtain the following assertion. Note that its formulation without a proof was given in [10, Th. 2].

Proposition 1. *Let* $\alpha : \mathbb{Z}_2 \longrightarrow Aut(C_r^*(\mathbb{Z} \times \mathbb{N}))$ *be the group homomorphism defined by*

$$\alpha_k(T_{(m,n)}) = \begin{cases} T_{(m,n)}, & \text{if } k = 0; \\ T_{(-m,n)}, & \text{if } k = 1, \end{cases}$$

where $m \in \mathbb{Z}$, $n \in \mathbb{N}$. *Then, there exists an isomorphism of* C^**-algebras*

$$C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times) \cong C_r^*(\mathbb{Z} \times \mathbb{N}) \rtimes_\alpha \mathbb{Z}_2.$$

Proof. Define the semidirect product $(\mathbb{Z} \times \mathbb{N}) \rtimes_\beta \mathbb{Z}_2$, where the action of the group $\mathbb{Z}_2$ on the semigroup $\mathbb{Z} \times \mathbb{N}$ is given by the formulas $\beta_0(m,n) = (m,n)$ and $\beta_1(m,n) = (-m,n)$ for all $m \in \mathbb{Z}$, $n \in \mathbb{N}$. It is easy to see we have the semigroup isomorphism

$$\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times \cong (\mathbb{Z} \times \mathbb{N}) \rtimes_\beta \mathbb{Z}_2 : (m,n) \mapsto \begin{cases} ((m,n),0), & \text{if } n > 0; \\ ((m,-n),1), & \text{if } n < 0, \end{cases}$$

where $m \in \mathbb{Z}$, $n \in \mathbb{Z}^\times$. Moreover, the homomorphisms α and β are connected with the formula $\alpha_k(T_{(m,n)}) = T_{\beta_k(m,n)}$, where $k \in \{0,1\}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$. Then, using Theorem 1 and the amenability of the group $\mathbb{Z}_2$, we obtain

$$C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times) \cong C_r^*((\mathbb{Z} \times \mathbb{N}) \rtimes_\beta \mathbb{Z}_2) \cong C_r^*(\mathbb{Z} \times \mathbb{N}) \rtimes_{\alpha,r} \mathbb{Z}_2 = C_r^*(\mathbb{Z} \times \mathbb{N}) \rtimes_\alpha \mathbb{Z}_2.$$

□

In [11], a relation between the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ and the infinite dihedral group was considered. We recall that the infinite dihedral group is the group $D_\infty := \mathbb{Z} \rtimes_\psi \mathbb{Z}_2$, where $\psi : \mathbb{Z}_2 \longrightarrow \mathrm{Aut}(\mathbb{Z})$ is the group homomorphism such that

$$\psi(0)(n) = n \quad \text{and} \quad \psi(1)(n) = -n$$

whenever $n \in \mathbb{Z}$. It is worth noting that Theorem 3 in [11] is a corollary of Theorem 1.

Finally, using Theorem 1, we prove the following statement. To this end, we introduce some additional notation. Let $\mathbb{N}$ stand for the multiplicative semigroup of natural numbers. By *tr* we denote both the trivial homomorphism $\mathrm{tr} : D_\infty \longrightarrow \mathrm{Aut}(C_r^*(\mathbb{N}))$ taking each element of D_∞ to the identity automorphism of the C^*-algebra $C_r^*(\mathbb{N})$ and the trivial action $\mathrm{tr} : D_\infty \longrightarrow \mathrm{Aut}(\mathbb{N})$ of the group D_∞ on the semigroup $\mathbb{N}$.

Proposition 2. *There exists an isomorphism of* C^**-algebras* $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times) \cong C_r^*(\mathbb{N}) \rtimes_{\mathrm{tr}} D_\infty$.

Proof. Take the Cartesian product $\mathbb{N} \times D_\infty$. Here, we treat $\mathbb{N} \times D_\infty$ as the semigroup with the coordinatewise binary operation. Obviously, we have the equality $\mathbb{N} \times D_\infty = \mathbb{N} \rtimes_{\mathrm{tr}} D_\infty$.

It is straightforward to verify that we have the isomorphism of the semigroups $\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times \cong \mathbb{N} \times D_\infty$ defined by

$$(m,n) \longmapsto \begin{cases} (n,(m,0)), & \text{if } n > 0; \\ (-n,(m,1)), & \text{if } n < 0; \end{cases}$$

whenever $m \in \mathbb{Z}$, $n \in \mathbb{Z}^\times$. Then, using Theorem 1 and the amenability of the group D_∞ (see, for example, [13, Section 1]), we obtain

$$C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times) \cong C_r^*(\mathbb{N} \times D_\infty) \cong C_r^*(\mathbb{N}) \rtimes_{\mathrm{tr},r} D_\infty = C_r^*(\mathbb{N}) \rtimes_{\mathrm{tr}} D_\infty,$$

as required. □

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