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On C^*-Algebra and $*$-Polynomial Relations

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Abstract—The note deals with categories whose objects are functions from sets to C^*-algebras and morphisms are $*$-homomorphisms of C^*-algebras making appropriate diagrams commutative. In the theory of universal C^*-algebras, such categories satisfying certain additional axioms are called C^*-relations. Those C^*-relations that determine universal C^*-algebras are said to be compact. In this note, we construct functors between compact C^*-relations. These functors arise from $*$-homomorphisms between universal C^*-algebras which are determined by compact C^*-relations. In the case when a functor is defined by an isomorphism of the universal C^*-algebras, we show that this functor is an isomorphism of compact C^*-relations. Moreover, we consider C^*-relations which are called $*$-polynomial relations associated with $*$-polynomial pairs. It is shown that every C^*-algebra is the universal C^*-algebra generated by a $*$-polynomial pair. As a consequence, we obtain that every compact C^*-relation is isomorphic to a $*$-polynomial relation.

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1. INTRODUCTION

The motivation for this note comes from the theory of universal C^*-algebras generated by sets of generators subject to relations (see [1–5]). An axiomatic approach to relations that correspond to universal C^*-algebras has been developed in [5]. In the framework of this approach, one deals with categories called C^*-relations. For a C^*-relation, objects are functions from a fixed set to C^*-algebras and morphisms are $*$-homomorphisms of C^*-algebras making appropriate diagrams commutative. Moreover, every C^*-relation satisfies certain axioms. Those C^*-relations that determine universal C^*-algebras are said to be compact (see [5], Section 2).

In this note we introduce functors acting between compact C^*-relations. Such a functor $\mathfrak{F}_\alpha : \mathcal{R}_2 \longrightarrow \mathcal{R}_1$ is constructed from a given $*$-homomorphism $\alpha : C^*(\mathcal{R}_1) \longrightarrow C^*(\mathcal{R}_2)$ between universal C^*-algebras $C^*(\mathcal{R}_1)$ and $C^*(\mathcal{R}_2)$ which are determined by compact C^*-relations $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively. It is shown that if α is an isomorphism of C^*-algebras, then the functor $\mathfrak{F}_\alpha$ is an isomorphism of C^*-relations. Further, we consider C^*-relations which are called $*$-polynomial relations associated with $*$-polynomial pairs. A polynomial pair (X, P) consists of a non-empty set X and a non-empty subset P of the free $*$-algebra $F(X)$ generated by X over the field of complex numbers. The objects of the $*$-polynomial relation $\mathcal{R}(X, P)$ associated with (X, P) are all functions f from the set X to C^*-algebras satisfying the following property: the set P is contained in the kernel of the unique $*$-homomorphism which is an extension of f to the free $*$-algebra $F(X)$. For two objects $f : X \longrightarrow A$ and $g : X \longrightarrow B$ in

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$\mathcal{R}(X, P)$, the morphisms from f to g are all $*$-homomorphisms of C^*-algebras of the form $\varphi : A \longrightarrow B$ such that $\varphi \circ f = g$. We prove that every C^*-algebra is a universal C^*-algebra determined by a $*$-polynomial relation. As an application of the above-mentioned results, we prove that every compact C^*-relation is isomorphic to a $*$-polynomial relation.

The note is organized as follows. It consists of Introduction and three more sections. Section 2 contains necessary notation and definitions from the theory of C^*-relations. In Section 3 we construct functors between compact C^*-relations. In Section 4 we consider $*$-polynomial relations associated with $*$-polynomial pairs and universal C^*-algebras generated by $*$-polynomial pairs.

2. PRELIMINARIES

Throughout this note we consider associative involutive algebras over the field of complex numbers. As usual, the symbol $*$ stands for involutions on algebras. The trivial algebra consisting of one zero element is denoted by 0.

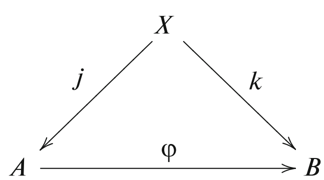
Let X be a non-empty set. We denote by $F(X)$ the free $*$-algebra of all $*$-polynomials in non-commuting variables generated by X. For a family $\{A_\lambda \mid \lambda \in \Lambda\}$ of C^*-algebras, we consider the direct product

$$\prod_{\lambda \in \Lambda} A_\lambda := \left\{ (a_\lambda) \,\Big|\, ||(a_\lambda)|| = \sup_{\lambda} ||a_\lambda|| < +\infty \right\}$$

which is a C^*-algebra with respect to the coordinatewise algebraic operations and the supremum norm.

Further, we recall necessary definitions from [5]. For the basic facts from the theory of categories and functors we refer the reader to the book [6].

For a given set X, *the null C^*-relation on X* is the category $\mathcal{F}_X$ whose objects are all functions of the form $j : X \longrightarrow A$, where A is a C^*-algebra. For two objects $j : X \longrightarrow A$ and $k : X \longrightarrow B$ in $\mathcal{F}_X$, a morphism from j to k is any $*$-homomorphism of C^*-algebras $\varphi : A \longrightarrow B$ making the diagram



(1)

commutative, i.e., $k = \varphi \circ j$.

A C^*-relation on X is a full subcategory $\mathcal{R}$ of $\mathcal{F}_X$ satisfying the following axioms:

C1: the function $X \longrightarrow 0$ is an object of $\mathcal{R}$;

C2: if $\varphi : A \longrightarrow B$ is an injective $*$-homomorphism of C^*-algebras, $f : X \longrightarrow A$ is a function and $\varphi \circ f$ is an object of $\mathcal{R}$, then f is an object of $\mathcal{R}$;

C3: if $\varphi : A \longrightarrow B$ is a $*$-homomorphism of C^*-algebras and $f : X \longrightarrow A$ is an object of $\mathcal{R}$, then $\varphi \circ f$ is an object of $\mathcal{R}$;

C4f: if $f_i : X \longrightarrow A_i$ is an object of $\mathcal{R}$ for every $i = 1, \ldots, n$, $n \in \mathbb{N}$, then the function

$$\prod_{i=1}^{n} f_i : X \longrightarrow \prod_{i=1}^{n} A_i$$

is an object of $\mathcal{R}$.

Objects of C^*-relations are also called *representations*.

A C^*-relation $\mathcal{R}$ on a set X is said to be *compact* if, in addition, the following condition is fulfilled:

C4: for any non-empty set Λ, if $f_\lambda : X \longrightarrow A_\lambda$ is an object of $\mathcal{R}$ for every $\lambda \in \Lambda$, then the function

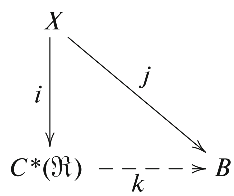
$$\prod_{\lambda\in\Lambda} f_\lambda : X \longrightarrow \prod_{\lambda\in\Lambda} A_\lambda$$

is also an object of $\mathcal{R}$.

The following statement is a reformulation of Theorem 2.10 from [5] (see also [2], Proposition 1.3.6; [3], Sect. 3.1; and [4], Sect. 1.4).

Proposition 1. *Let $\mathcal{R}$ be a C^*-relation on a set X. Then, $\mathcal{R}$ is compact if and only if there exists an initial object in $\mathcal{R}$.*

In what follows, for a compact C^*-relation $\mathcal{R}$ on a set X, we consider an initial object $i : X \longrightarrow A$ of $\mathcal{R}$. The C^*-algebra A is denoted by $C^*(\mathcal{R})$. Thus, for every representation $j : X \longrightarrow B$ of $\mathcal{R}$ there exists a unique $*$-homomorphism of C^*-algebras $k : C^*(\mathcal{R}) \longrightarrow B$ such that the diagram



is commutative, i.e., $j = k \circ i$. In this case, we denote the $*$-homomorphism k by $\overline{j}$. Obviously, we have

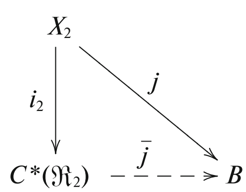
$$\overline{j} = \overline{\overline{j} \circ i}. \tag{2}$$

3. FUNCTORS BETWEEN COMPACT C^*-RELATIONS

Throughout this section, $\mathcal{R}_1$ and $\mathcal{R}_2$ are compact C^*-relations on sets X_1 and X_2 respectively. Let $i_t : X_t \longrightarrow C^*(\mathcal{R}_t)$ be an initial object in the category $\mathcal{R}_t$, where $t = 1, 2$.

Assume that we are given a $*$-homomorphism of C^*-algebras $\alpha : C^*(\mathcal{R}_1) \longrightarrow C^*(\mathcal{R}_2)$. Now, we are going to construct a covariant functor $\mathfrak{F}_\alpha : \mathcal{R}_2 \longrightarrow \mathcal{R}_1$ associated with the $*$-homomorphism α.

Firstly, we take an object $j : X_2 \longrightarrow B$ of the category $\mathcal{R}_2$. To define the image of j under the action of the functor $\mathfrak{F}_\alpha : \mathcal{R}_2 \longrightarrow \mathcal{R}_1$, we take the $*$-homomorphism of C^*-algebras $\overline{j} : C^*(\mathcal{R}_2) \longrightarrow B$ making the diagram

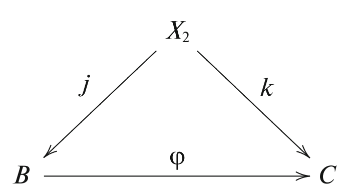


(3)

commutative and consider the function $\overline{j} \circ \alpha \circ i_1 : X_1 \longrightarrow B$. By the axiom **C3**, the composition $\overline{j} \circ \alpha \circ i_1$ is an object of the category $\mathcal{R}_1$. We define $\mathfrak{F}_\alpha(j)$ by

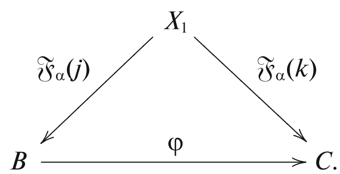
$$\mathfrak{F}_\alpha(j) := \overline{j} \circ \alpha \circ i_1. \tag{4}$$

Secondly, to define the arrow function for $\mathfrak{F}_\alpha$, we take a morphism $\varphi : B \longrightarrow C$ in $\mathcal{R}_2$ making the diagram



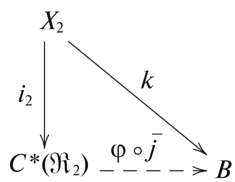
(5)

commutative. Then, we can consider the objects $\mathfrak{F}_\alpha(j) : X_1 \longrightarrow B$ and $\mathfrak{F}_\alpha(k) : X_1 \longrightarrow C$ in the category $\mathcal{R}_1$ and the diagram



(6)

We claim that it is commutative. Indeed, using the commutativity of the diagrams (3) and (5), we get $\varphi \circ \overline{j} \circ i_2 = \varphi \circ j = k$, which means that the diagram



is commutative. Hence, by definition of the $*$-homomorphism $\overline{k} : C^*(\mathcal{R}_2) \longrightarrow C$, we obtain the equality

$$\varphi \circ \overline{j} = \overline{k}. \tag{7}$$

Further, by (7), we have

$$\varphi \circ \mathfrak{F}_\alpha(j) = \varphi \circ \overline{j} \circ \alpha \circ i_1 = \overline{k} \circ \alpha \circ i_1 = \mathfrak{F}_\alpha(k),$$

which means the commutativity of the diagram (6), as claimed. Therefore, the $*$-homomorphism φ can be viewed as a morphism from $\mathfrak{F}_\alpha(j)$ to $\mathfrak{F}_\alpha(k)$ in the category $\mathcal{R}_1$, and we can put

$$\mathfrak{F}_\alpha(\varphi) := \varphi. \tag{8}$$

It is straightforward to check that $\mathfrak{F}_\alpha$ preserves the identity morphisms and the compositions of morphisms. That is, we have

$$\mathfrak{F}_\alpha(1_j) = 1_{\mathfrak{F}_\alpha(j)} \quad \text{and} \quad \mathfrak{F}_\alpha(\psi \circ \varphi) = \mathfrak{F}_\alpha(\psi) \circ \mathfrak{F}_\alpha(\varphi)$$

whenever j is an object of the category $\mathcal{R}_2$, $1_j : j \longrightarrow j$ is the identity arrow in $\mathcal{R}_2$, φ and ψ are morphisms in $\mathcal{R}_2$ such that the composite $\psi \circ \varphi$ is defined in $\mathcal{R}_2$. Thus, the formulas (4) and (8) define the object and the arrow functions respectively, and we have constructed the functor $\mathfrak{F}_\alpha : \mathcal{R}_2 \longrightarrow \mathcal{R}_1$, as desired.

Furthermore, if $\alpha : C^*(\mathcal{R}_1) \longrightarrow C^*(\mathcal{R}_2)$ is an isomorphism of C^*-algebras, then we have the functors $\mathfrak{F}_\alpha : \mathcal{R}_2 \longrightarrow \mathcal{R}_1$ and $\mathfrak{F}_{\alpha^{-1}} : \mathcal{R}_1 \longrightarrow \mathcal{R}_2$, where $\alpha^{-1} : C^*(\mathcal{R}_2) \longrightarrow C^*(\mathcal{R}_1)$ is the inverse of the $*$-homomorphism α.

We claim that both composites $\mathfrak{F}_{\alpha^{-1}} \circ \mathfrak{F}_\alpha$ and $\mathfrak{F}_\alpha \circ \mathfrak{F}_{\alpha^{-1}}$ are the identity functors on the categories $\mathcal{R}_2$ and $\mathcal{R}_1$, respectively. Indeed, it suffices to prove that

$$\mathfrak{F}_{\alpha^{-1}} \circ \mathfrak{F}_\alpha = 1_{\mathcal{R}_2}, \tag{9}$$

where $1_{\mathcal{R}_2}$ is the identity functor on $\mathcal{R}_2$. To show this, we take an object j in the category $\mathcal{R}_2$. Using the equality (2) and the commutative diagram (3), we get

$$\mathfrak{F}_{\alpha^{-1}} \circ \mathfrak{F}_\alpha(j) = \mathfrak{F}_{\alpha^{-1}}(\overline{j} \circ \alpha \circ i_1) = \overline{\overline{j} \circ \alpha \circ i_1} \circ \alpha^{-1} \circ i_2 = \overline{j} \circ \alpha \circ \alpha^{-1} \circ i_2 = \overline{j} \circ i_2 = j.$$

Of course, we also have $\mathfrak{F}_{\alpha^{-1}} \circ \mathfrak{F}_\alpha(\varphi) = \varphi$ whenever φ is a morphism in the category $\mathcal{R}_2$. Hence, the equality (9) holds, as claimed. Thus, the functors $\mathfrak{F}_\alpha : \mathcal{R}_2 \longrightarrow \mathcal{R}_1$ and $\mathfrak{F}_{\alpha^{-1}} : \mathcal{R}_1 \longrightarrow \mathcal{R}_2$ are isomorphisms of the categories. Summarizing the above observations, we have

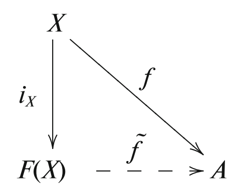
Theorem 1. *Let $X_1 \longrightarrow C^*(\mathcal{R}_1)$ and $X_2 \longrightarrow C^*(\mathcal{R}_2)$ be initial objects in compact C^*-relations $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. Then, every $*$-homomorphism of C^*-algebras $\alpha : C^*(\mathcal{R}_1) \longrightarrow C^*(\mathcal{R}_2)$ generates the covariant functor $\mathfrak{F}_\alpha : \mathcal{R}_2 \longrightarrow \mathcal{R}_1$. Moreover, if α is an isomorphism of C^*-algebras, then $\mathfrak{F}_\alpha$ is an isomorphism of the categories $\mathcal{R}_2$ and $\mathcal{R}_1$.*

4. $*$-POLYNOMIAL RELATIONS

This section deals with $*$-polynomial relations associated with $*$-polynomial pairs. We begin with necessary definitions.

Definition 1. If X is a non-empty set and P is a non-empty subset of the free $*$-algebra $F(X)$, then the pair (X, P) is said to be $*$-*polynomial*.

To introduce an additional notation, we recall the universal property of the free $*$-algebra $F(X)$. Namely, for every mapping $f : X \longrightarrow A$ from a set X to a $*$-algebra A there exists a unique $*$-homomorphism $\tilde{f} : F(X) \longrightarrow A$ which is an extension of f to $F(X)$. That is, for the embedding $i_X : X \longrightarrow F(X)$, the diagram



is commutative.

Definition 2. A function $f : X \longrightarrow A$ from a set X to a C^*-algebra A is called a *representation* of a $*$-polynomial pair (X, P) provided that $P \subset Ker\tilde{f}$, where $Ker\tilde{f}$ is the kernel of the $*$-homomorphism $\tilde{f} : F(X) \longrightarrow A$.

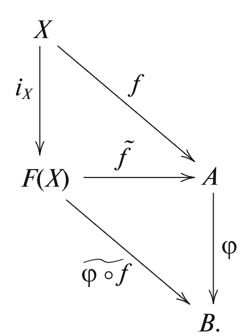
Let (X, P) be a $*$-polynomial pair. We consider the category $\mathcal{R}(X, P)$ whose objects are all representations of the $*$-polynomial pair (X, P). For representations $j : X \longrightarrow A$ and $k : X \longrightarrow B$ of (X, P), a morphism from j to k is any $*$-homomorphism of C^*-algebras $\varphi : A \longrightarrow B$ such that the diagram (1) is commutative. Thus, $\mathcal{R}(X, P)$ is a full subcategory of the null C^*-relation $\mathcal{F}_X$ on X.

Definition 3. The category $\mathcal{R}(X, P)$ is called a $*$-*polynomial relation associated with a* $*$-*polynomial pair* (X, P).

In what follows, we treat certain properties of $*$-polynomial relations.

Proposition 2. *For any* $*$-*polynomial pair* (X, P)*, the* $*$-*polynomial relation* $\mathcal{R}(X, P)$ *is a* C^*-*relation.*

Proof. The axiom **C1** is obviously satisfied in the category $\mathcal{R}(X, P)$. Further, let $\varphi : A \longrightarrow B$ be a $*$-homomorphism of C^*-algebras and $f : X \longrightarrow A$ be a function. Consider the diagram



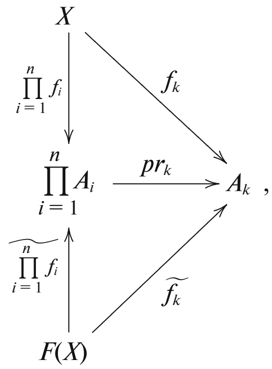
Using the universal property of the free $*$-algebra $F(X)$, we get the equality $\widetilde{\varphi \circ f} = \varphi \circ \tilde{f}$. So we have

$$\ker \tilde{f} \subset \ker(\varphi \circ \tilde{f}) = \ker(\widetilde{\varphi \circ f}).$$

This implies the axiom **C3**.

If φ is an injective $*$-homomorphism, then one has the equality $\ker \tilde{f} = \ker \widetilde{\varphi \circ f}$. Hence, the axiom **C2** holds.

Finally, let $f_i : X \longrightarrow A_i$ be a function for every $i = 1, \dots, n$, where $n \in \mathbb{N}$. For each $k \in \{1, \dots, n\}$, we consider the diagram



where pr_k denotes the natural projection. Since $pr_k \circ \widetilde{\prod_{i=1}^n f_i} \circ i_X = pr_k \circ \prod_{i=1}^n f_i = f_k$, one has the equality $pr_k \circ \widetilde{\prod_{i=1}^n f_i} = \widetilde{f_k}$. Therefore, we get $\prod_{i=1}^n \widetilde{f_i} = \widetilde{\prod_{i=1}^n f_i}$ as well as the equalities

$$\bigcap_{i=1}^{n} \ker \widetilde{f_i} = \ker \left(\prod_{i=1}^{n} \widetilde{f_i} \right) = \ker \left(\widetilde{\prod_{i=1}^{n} f_i} \right).$$

This implies the axiom **C4f**. Thus, the category $\mathcal{R}(X, P)$ is a C^*-relation. □

The following is an example of a $*$-polynomial relation associated with a $*$-polynomial pair which is not a compact C^*-relation.

Example. Let $X = \{x\}$ be a one-element set and $P = \{xx^* - x^*x\}$ be the subset of the free $*$-algebra $F(X)$ consisting of the single polynomial. Consider the $*$-polynomial relation $\mathcal{R}(X, P)$ associated with the $*$-polynomial pair (X, P). Let A be a C^*-algebra and $a \in A$ be a non-zero normal element. For each $n \in \mathbb{N}$ we take the object f_n of the category $\mathcal{R}(X, P)$ defined by

$$f_n : X \longrightarrow A : x \longmapsto na.$$

Since $\sup_{n\in\mathbb{N}} ||f_n(x)|| = +\infty$, the axiom **C4** is false in the $*$-polynomial relation $\mathcal{R}(X, P)$. That is, the category $\mathcal{R}(X, P)$ is not a compact C^*-relation, as desired.

Definition 4. If a $*$-polynomial relation $\mathcal{R}(X, P)$ has an initial object $i : X \longrightarrow A$, then the C^*-algebra A is called the *universal C^*-algebra* generated by the $*$-polynomial pair (X, P) and is denoted by $C^*(X, P)$.

The following theorem states that every C^*-algebra is a *universal C^*-algebra* generated by a $*$-polynomial pair.

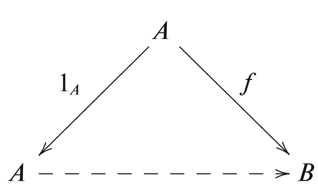
Theorem 2. *For every C^*-algebra A there exists a $*$-polynomial pair (X, P) such that $A = C^*(X, P)$.*

Proof. Let us take the identity mapping $1_A : A \longrightarrow A$ and its extension $\tilde{1}_A : F(A) \longrightarrow A$. We have

$$\tilde{1}_A \circ i_A = 1_A. \tag{10}$$

We prove that A is the universal C^*-algebra generated by the $*$-polynomial pair $(A, Ker\tilde{1}_A)$.

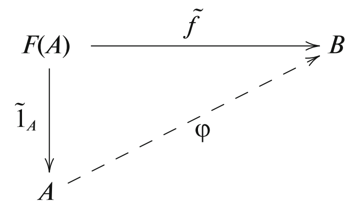
More precisely, we claim that $1_A : A \longrightarrow A$ is an initial object in the $*$-polynomial relation $\mathcal{R}(A, Ker\tilde{1}_A)$. Indeed, let us take a representation $f : A \longrightarrow B$ of the $*$-polynomial pair $(A, Ker\tilde{1}_A)$. We need to show that there is a unique $*$-homomorphism from the C^*-algebra A to the C^*-algebra B such that the diagram



(11)

is commutative. Informally, we have to show that f is a $*$-homomorphism of C^*-algebras or, more generally, that every object in the category $\mathcal{R}(A, Ker\tilde{1}_A)$ is a $*$-homomorphism of C^*-algebras.

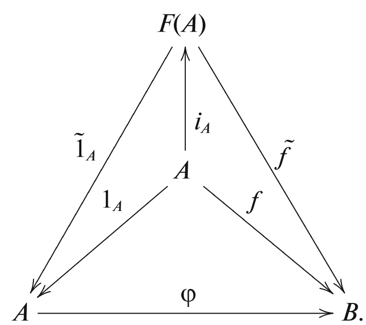
Since f is a representation of the $*$-polynomial pair $(A, Ker\tilde{1}_A)$, we have $Ker\tilde{1}_A \subset Ker\tilde{f}$. This inclusion guarantees that there exists a unique $*$-homomorphism $\varphi : A \longrightarrow B$ of C^*-algebras making the diagram



commutative, i.e., we have

$$\tilde{f} = \varphi \circ \tilde{1}_A. \tag{12}$$

Further, we consider the diagram



Using the universal property of the $*$-algebra $F(A)$ together with the equalities (12) and (10), we get

$$f = \tilde{f} \circ i_A = \varphi \circ \tilde{1}_A \circ i_A = \varphi \circ 1_A.$$

This means that we have the unique $*$-homomorphism, namely, $\varphi = f$, making the diagram (11) commutative. Thus, $1_A : A \longrightarrow A$ is an initial object in the $*$-polynomial relation $\mathcal{R}(A, Ker\tilde{1}_A)$, as claimed. This completes the proof. □

Corollary 1. *For every C^*-algebra A the $*$-polynomial relation $\mathcal{R}(A, Ker\tilde{1}_A)$ is a compact C^*-relation.*

Proof. Combining Propositions 1, 2 and the fact that the identity function $1_A : A \longrightarrow A$ is an initial object in the category $\mathcal{R}(A, Ker\tilde{1}_A)$, we get the statement. □

Finally, we use the previous results to prove

Theorem 3. *Every compact C^*-relation is isomorphic to a $*$-polynomial relation.*

Proof. Let $\mathcal{R}$ be a compact C^*-relation on a set X and $i : X \longrightarrow C^*(\mathcal{R})$ be its initial object.

By Theorem 2, we have $C^*(\mathcal{R}) = C^*(C^*(\mathcal{R}), Ker\tilde{1}_{C^*(\mathcal{R})})$, i.e., $C^*(\mathcal{R})$ is the universal C^*-algebra generated by the $*$-polynomial pair $(C^*(\mathcal{R}), Ker\tilde{1}_{C^*(\mathcal{R})})$, and the identity mapping $1_{C^*(\mathcal{R})} : C^*(\mathcal{R}) \longrightarrow C^*(\mathcal{R})$ is an initial object in the $*$-polynomial relation $\mathcal{R}(C^*(\mathcal{R}), Ker\tilde{1}_{C^*(\mathcal{R})})$ associated with $(C^*(\mathcal{R}), Ker\tilde{1}_{C^*(\mathcal{R})})$. Corollary 1 guarantees that the $*$-polynomial relation $\mathcal{R}(C^*(\mathcal{R}), Ker\tilde{1}_{C^*(\mathcal{R})})$ is a compact C^*-relation.

Since the identity $*$-homomorphism $1_{C^*(\mathcal{R})} : C^*(\mathcal{R}) \longrightarrow C^*(\mathcal{R})$ is an isomorphism of C^*-algebras, Theorem 1 yields the functor

$$\mathfrak{F}_{1_{C^*(\mathcal{R})}} : \mathcal{R}(C^*(\mathcal{R}), Ker\tilde{1}_{C^*(\mathcal{R})}) \longrightarrow \mathcal{R},$$

which is an isomorphism of compact C^*-relations. The proof is complete. □

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