ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2023, Vol. 44, No. 4, pp. 1332–1340.

A Semigroup C^*-Algebra Related To the Infinite Dihedral Group

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(Submitted by F. G. Avhadiev)

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Received November 20, 2022; revised December 6, 2022; accepted December 31, 2022

Abstract—The paper deals with the reduced semigroup C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ for the semidirect product $\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times$ of the additive group $\mathbb{Z}$ of all integers and the multiplicative semigroup $\mathbb{Z}^\times$ of integers without zero. The semigroup homomorphism φ acts from $\mathbb{Z}^\times$ to the endomorphism semigroup of $\mathbb{Z}$ as follows. It takes every positive integer to the identity endomorphism of $\mathbb{Z}$ and every negative integer to the inversion of $\mathbb{Z}$. The purpose of the paper is to demonstrate that the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ and the infinite dihedral group D_∞ are closely related. To this end, we prove three results. Firstly, we show that the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ is topologically graded over the group D_∞. In order to obtain this result, we use a general method for constructing topological gradings. Previously, this method was proposed in the study of extensions of semigroups and the reduced semigroup C^*-algebras associated with such extensions. As a consequence, the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ has a countable family of Fourier coefficients which is indexed by the elements of D_∞. Secondly, we construct a covariant representation of a non-commutative dynamical system defined by means of the group D_∞ into the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$. Thirdly, we establish an isomorphism between the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ and a crossed product of its C^*-subalgebra by the group D_∞.

DOI: 10.1134/S199508022304011X

Keywords and phrases: *covariant representation, crossed product of a C^*-algebra by a group, dynamical system, infinite dihedral group, reduced semigroup C^*-algebra, semidirect product of semigroups, topologically graded C^*-algebra.*

1. INTRODUCTION

The main object of our study is the reduced semigroup C^*-algebra for a semidirect product of the additive group $\mathbb{Z}$ of all integers and the multiplicative semigroup $\mathbb{Z}^\times$ of integers without zero.

The reduced semigroup C^*-algebras are very natural objects. They are generated by the left regular representations of semigroups with the cancelation property. The start in studying these algebras was made by Coburn [1, 2] who considered the reduced semigroup C^*-algebra for the additive semigroup of the non-negative integers. Douglas [3] investigated the case of subsemigroups in the additive group of the real numbers. Murphy [4, 5] generalized the results from [1–3] to the case of the reduced semigroup C^*-algebras for the positive cones in ordered groups. These authors proved that the isometric representations of the specified semigroups have the universal property.

For extensive literature and history of the study of semigroup C^*-algebras, the reader is referred, for example, to [6] and the references therein.

The present paper is concerned with the reduced semigroup C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ for the semidirect product $\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times$ of $\mathbb{Z}$ and $\mathbb{Z}^\times$ relative to the semigroup homomorphism φ from $\mathbb{Z}^\times$ to the

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endomorphism semigroup of $\mathbb{Z}$. The homomorphism φ takes every positive integer to the identity endomorphism of $\mathbb{Z}$ and every negative integer to the inversion of $\mathbb{Z}$.

The semigroup C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ was treated in [7, 8]. It was shown that this algebra provides an example of C^*-algebra which is very interesting to study. In particular, it is closely related to the finite dihedral groups. For instance ([7], Theorem 3.1), the semigroup C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ is topologically graded over the dihedral group D_p for every $p \geq 2$. Here, it is worth noting that the question about the existence of a topological grading for a semigroup C^*-algebra is connected with the problem on constructions of normal extensions for semigroups (see [7—10]).

In this paper, we demonstrate that the reduced semigroup C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ is also closely related to the infinite dihedral group D_∞. Firstly, we prove that this C^*-algebra can be topologically graded over the group D_∞. This result sheds light on the structure of the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$. Furthermore, it implies the existence of a countable family of non-commutative Fourier coefficients which is indexed by the elements of the group D_∞. The crucial role in constructing the topological grading for the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ belongs to the notion of the σ-index of an operator monomial which was introduced by the second-named author in [11, 12]. Secondly, we construct a covariant representation of a non-commutative dynamical system into the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$. This dynamical system is a triple $(\mathfrak{B}, D_\infty, tr)$ consisting of a C^*-subalgebra $\mathfrak{B}$ in the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$, the group D_∞ and the trivial action tr of D_∞ on $\mathfrak{B}$. The C^*-subalgebra $\mathfrak{B}$ is generated by a countable family of isometries. Thirdly, we show that the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ is isomorphic to the crossed product of the C^*-subalgebra $\mathfrak{B}$ by the group D_∞ relative to the trivial action tr. To establish this isomorphism, we use the amenability of the infinite dihedral group D_∞ as well as some properties of the spatial and the maximal tensor products of C^*-algebras.

Now we recall the definition of the reduced semigroup C^*-algebra.

Let S be a discrete left cancelative semigroup. As usual, the symbol $l^2(S)$ stands for the Hilbert space of all square summable complex valued functions on S. For every $a \in S$, we denote by e_a the function in $l^2(S)$ which is defined as follows: $e_a(b) = 1$, if $a = b$, and $e_a(b) = 0$, if $a \neq b$, where $b \in S$. Then, the set of functions $\{e_a \mid a \in S\}$ is an orthonormal basis in the Hilbert space $l^2(S)$.

In the C^*-algebra of all bounded linear operators $B(l^2(S))$ on the Hilbert space $l^2(S)$, we define the C^*-subalgebra $C_r^*(S)$ generated by the set of isometries $\{T_a \mid a \in S\}$, where $T_a(e_b) = e_{ab}$ for $a, b \in S$. It is called *the reduced semigroup C^*-algebra*. The identity element in this algebra is denoted by I.

In the same way, *the reduced group C^*-algebra $C_r^*(G)$* is defined for a discrete group G. Namely, it is the C^*-subalgebra in $B(l^2(G))$ which is generated by the set of the unitary operators $\{S_g \mid g \in G\}$, where $S_g(f)(h) = f(g^{-1}h)$ whenever $f \in l^2(G)$ and $g, h \in G$. Of course, one has $S_g^* = S_{g^{-1}}$, where $g \in G$.

We recall that there is another very important C^*-algebra defined for G. It is *the full group C^*-algebra $C^*(G)$*. In general, the group algebras $C_r^*(G)$ and $C^*(G)$ are not isomorphic. The amenability of G guarantees the existence of an isomorphism between the group C^*-algebras $C_r^*(G)$ and $C^*(G)$. For details, the reader is referred to ([13], Ch. IV, Sec. 7).

We notice that the *C^*-crossed product* is a generalization of the full group C^*-algebra for G. As is well known, the C^*-algebra $C^*(G)$ is the crossed product of the field of complex numbers by the group G. The construction of the crossed product $\mathcal{A} \rtimes_\alpha \mathcal{G}$ of a C^*-algebra $\mathcal{A}$ by a locally compact group $\mathcal{G}$ is contained, for instance, in [14]. The C^*-crossed products are well studied and widely used in the theory of operator algebras and mathematical physics.

The paper is organized as follows. It consists of introduction and two sections. The first section deals with the proof of Theorem 1 on a topological grading of the reduced semigroup C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ over the infinite dihedral group D_∞. The second section is devoted to the results on non-commutative dynamical systems and the C^*-crossed products. To obtain these results we firstly study the relations between the generating elements of the reduced semigroup C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$. Theorem 2 states the existence of a covariant representation for the dynamical system $(\mathfrak{B}, D_\infty, \mathrm{tr})$ into the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$. In Theorem 3 we construct an isomorphism between the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ and the crossed product of the C^*-subalgebra $\mathfrak{B}$ by the infinite dihedral group D_∞.

2. TOPOLOGICAL GRADING FOR THE C^*-ALGEBRA $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ OVER THE DIHEDRAL GROUP D_∞

In this Section we construct the reduced semigroup C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ and show that it is topologically graded over the infinite dihedral group D_∞.

As usual, we denote by $\mathbb{Z}$ the additive group of all integers. Let $\mathbb{Z}^\times$ be the multiplicative semigroup $\mathbb{Z} \setminus \{0\}$ and let $\varphi : \mathbb{Z}^\times \longrightarrow \mathrm{End}(\mathbb{Z})$ be the semigroup homomorphism from $\mathbb{Z}^\times$ into the semigroup of endomorphisms of the group $\mathbb{Z}$ given by

$$\varphi(m) := \begin{cases} id, & \text{if } m > 0; \\ inv, & \text{if } m < 0, \end{cases}$$

where $m \in \mathbb{Z}^\times$, the symbol id stands for the identity endomorphism and inv means the inversion, that is, $inv(n) = -n$ whenever $n \in \mathbb{Z}$. We consider the semidirect product of $\mathbb{Z}$ and $\mathbb{Z}^\times$ with respect to φ which is denoted by $\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times$. It is a semigroup with respect to the multiplication defined by

$$(m, n)(k, l) = (m + \varphi(n)(k), nl), \tag{1}$$

where $m, k \in Z,\ n, l \in Z^\times$. It is straightforward to verify that $\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times$ is a semigroup with the cancelation property and the unit $(0, 1)$.

The reduced semigroup C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ of the semidirect product $\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times$ is studied in [7, 8]. We recall that this algebra is generated by the set of isometries $\{T_{(m,n)} \mid m \in \mathbb{Z}, n \in \mathbb{Z}^\times\}$ in the C^*-algebra $B(l^2(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times))$. As was mentioned above, we shall construct a topological grading for the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$.

For the sake of completeness, we now give the definitions of graded and topologically graded C^*-algebras. For details we refer the reader to the book [15, §§16.2, 19.2].

Let G be a group. A C^*-algebra $\mathfrak{A}$ is said to be *G-graded* if there exists a family of linearly independent closed subspaces $\{\mathfrak{A}_g\}_{g\in G}$ in $\mathfrak{A}$ such that the following conditions are satisfied:

1) $\mathfrak{A}_g\mathfrak{A}_h \subset \mathfrak{A}_{gh}$ whenever $g, h \in G$;

2) $\mathfrak{A}_g^* = \mathfrak{A}_{g^{-1}}$ for every $g \in G$;

3) $\mathfrak{A} = \overline{\bigoplus_{g\in G} \mathfrak{A}_g}$.

The family of Banach spaces $\{\mathfrak{A}_g\}_{g\in G}$ is called a *C^*-algebraic bundle*, or *a Fell bundle*, over the group G. It is worth noting that, for the identity element e of G, the space $\mathfrak{A}_e$ is a C^*-subalgebra in the C^*-algebra $\mathfrak{A}$.

A G-graded C^*-algebra $\mathfrak{A}$ with the Fell bundle $\{\mathfrak{A}_g\}_{g\in G}$ is said to be *topologically graded* if there exists a contractive linear operator

$$F : \mathfrak{A} \longrightarrow \mathfrak{A}_e$$

which coincides with the identity operator on $\mathfrak{A}_e$ and vanishes on each subspace $\mathfrak{A}_g$, where $g \in G, g \neq e$. In this case, the C^*-algebra $\mathfrak{A}$ possesses the Fourier coefficients for all $g \in G$ (see [15, § 19.6]). We recall that a contractive linear operator

$$F_g : \mathfrak{A} \longrightarrow \mathfrak{A}_g, \quad g \in G,$$

is called a Fourier coefficient for $\mathfrak{A}$ if $F_g(A) = A_g$ for each finite sum $A = \sum_{h\in G} A_h$.

Throughout this paper, the symbol D_∞ stands for the infinite dihedral group. It can be defined as the semidirect product of groups $D_\infty := \mathbb{Z} \rtimes_\psi \mathbb{Z}_2$, where $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ is the cyclic group of order two, $\psi : \mathbb{Z}_2 \longrightarrow \mathrm{Aut}(\mathbb{Z})$ is the group homomorphism from $\mathbb{Z}_2$ to the automorphism group of $\mathbb{Z}$ such that

$$\psi(0)(n) = n \quad \text{and} \quad \psi(1)(n) = -n$$

whenever $n \in \mathbb{Z}$.

Let us define the mapping σ as follows

$$\sigma : Z \rtimes_\varphi Z^\times \longrightarrow D_\infty : (m, n) \longmapsto \begin{cases} (m, 0), & \text{if } n > 0; \\ (m, 1), & \text{if } n < 0. \end{cases} \tag{2}$$

It is easily seen that σ is a surjective semigroup homomorphism.

In the sequel, we shall need a general method for constructing a topological grading over a group G for the reduced semigroup C^*-algebra $C_r^*(S)$ under the assumption that there exists a surjective semigroup homomorphism from a semigroup S onto G. This method was proposed in [11, 12]. It is based on the notion of the σ-index of an operator monomial (see details in [12]).

Further, we give a brief description of this method.

Let S and G be a semigroup and a group respectively. Assume that there exists a surjective semigroup homomorphism $\sigma : S \longrightarrow G$. We treat the free semigroup $\mathcal{F}(S)$ of *monomials*

$$W_{\overline{a}} := T_{a_1}^{i_1} T_{a_2}^{i_2} ... T_{a_k}^{i_k}, \tag{3}$$

where $\overline{a} = (a_1, ..., a_k)$ is an element of the Cartesian product of k copies of the semigroup S, $i_1, ..., i_k \in \{-1, 1\}, k \in \mathbb{N}$.

Let the mapping ind : $\mathcal{F}(S) \longrightarrow G$ take each monomial (3) to the element ind $(W_{\overline{a}})$ of the group G given by

$$\text{ind } (W_{\overline{a}}) = \sigma(a_1)^{i_1} \sigma(a_2)^{i_2} ... \sigma(a_k)^{i_k}.$$

One can easily verify that the mapping ind is a surjective semigroup homomorphism. The image ind $(W_{\overline{a}})$ of a monomial $W_{\overline{a}}$ under this homomorphism is called *the σ-index of the monomial $W_{\overline{a}}$*.

For every monomial (3), we have the operator $\widehat{W_{\overline{a}}} \in B(l^2(S))$ defined by $\widehat{W_{\overline{a}}} := \widehat{T}_{a_1}^{i_1} \widehat{T}_{a_2}^{i_2} ... \widehat{T}_{a_k}^{i_k}$, where $\widehat{T}_a^1 := T_a$ and $\widehat{T}_a^{-1} := T_a^*$. The operator $\widehat{W_{\overline{a}}}$ is called *an operator monomial*.

In ([12], Lemma 1), it is shown that for two monomials $W_{\overline{a}}$ and $W_{\overline{b}}$ the condition $\widehat{W_{\overline{a}}} = \widehat{W_{\overline{b}}}$ implies the equality ind $(W_{\overline{a}}) =$ ind $(W_{\overline{b}})$. As a consequence, the notion of *the σ-index of an operator monomial* is well-defined.

For every $g \in G$, the symbol $\mathfrak{A}_g$ stands for the subspace in the C^*-algebra $C_r^*(S)$ which is the closure of the linear span of all operator monomials with the σ-index g.

According to ([12], Theorem 2), this method yields a topological grading for the reduced semigroup C^*-algebra $C_r^*(S)$. In other words, the family of Banach spaces $\{\mathfrak{A}_g \mid g \in G\}$ satisfies the above-mentioned properties.

In the similar way, we obtain the following result.

Theorem 1. *The reduced semigroup C^*-algebra $C_r^*(Z \rtimes_\varphi Z^\times)$ is topologically graded over the infinite dihedral group D_∞.*

Proof. Indeed, we are given the surjective semigroup homomorphism

$$\sigma : Z \rtimes_\varphi Z^\times \longrightarrow D_\infty$$

defined by (2). It remains to apply the method described above and ([12], Theorem 2). □

Corollary 1. *Let the surjective semigroup homomorphism σ be defined by (2). For every $d \in D_\infty$ let $\mathfrak{A}_d$ denote the Banach subspace in the C^*-algebra $C_r^*(Z \rtimes_\varphi Z^\times)$ generated by all operator monomials with the σ-index d. Then, there exists a Fourier coefficient*

$$F_d : C_r^*(Z \rtimes_\varphi Z^\times) \longrightarrow \mathfrak{A}_d.$$

3. REPRESENTATION OF THE C^*-ALGEBRA $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ AS A CROSSED PRODUCT

In this section we show that the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ can be represented as a crossed product of its subalgebra by the infinite dihedral group D_∞.

We begin by recalling the necessary notions concerning the crossed products of the unital C^*-algebras by the discrete groups. The definition of the crossed product of an arbitrary C^*-algebra by a locally compact group is contained, for example, in [14]. It is worth noting that the author of [16] treats the crossed products in the terms of the universal property.

Let $\mathcal{A}$ be a unital C^*-algebra, G be a discrete group and $\alpha : G \longrightarrow \text{Aut}(\mathcal{A})$ be a homomorphism of groups. The triple $(\mathcal{A}, G, \alpha)$ is called *a dynamical system*.

A covariant representation of the dynamical system $(\mathcal{A}, G, \alpha)$ is a pair (π, u) consisting of a representation $\pi : \mathcal{A} \longrightarrow B(H)$ and a unitary representation $u : G \longrightarrow B(H)$ for a Hilbert space H such that

$$\pi(\alpha_g(a)) = u(g)\pi(a)u(g)^*$$

for all $a \in \mathcal{A}$ and $g \in G$ ([14], pp. 43–44). Certainly, one can say about a covariant representation of a dynamical system into a C^*-algebra.

The crossed product of $\mathcal{A}$ *by* G is a triple $(\mathcal{A} \rtimes_\alpha G, i_\mathcal{A}, i_G)$, where $\mathcal{A} \rtimes_\alpha G$ is a C^*-algebra, $i_\mathcal{A} : \mathcal{A} \longrightarrow \mathcal{A} \rtimes_\alpha G$ is a unital $*$-homomorphism of C^*-algebras, $i_G : G \longrightarrow U(\mathcal{A} \rtimes_\alpha G)$ is a group homomorphism from the group G to the group $U(\mathcal{A} \rtimes_\alpha G)$ of unitary elements of $\mathcal{A} \rtimes_\alpha G$ satisfying the following conditions

1) $i_\mathcal{A}(\alpha_g(a)) = i_G(g)i_\mathcal{A}(a)i_G(g)^*$ whenever $a \in \mathcal{A}$, $g \in G$;

2) for every covariant representation (π, u) of the dynamical system $(\mathcal{A}, G, \alpha)$ there exists a unital representation $\pi \rtimes u : \mathcal{A} \rtimes_\alpha G \longrightarrow B(H)$ such that
$$(\pi \rtimes u) \circ i_\mathcal{A} = \pi \quad \text{and} \quad (\pi \rtimes u) \circ i_G = u;$$

3) the C^*-algebra $\mathcal{A} \rtimes_\alpha G$ is generated by the set $\{i_\mathcal{A}(a) | a \in \mathcal{A}\} \cup \{i_G(g) | g \in G\}$.

The C^*-algebra $\mathcal{A} \rtimes_\alpha G$ itself is also called the crossed product of $\mathcal{A}$ by G.

It is worth noting that for every dynamical system $(\mathcal{A}, G, \alpha)$ there exists a crossed product. Moreover, it is unique up to isomorphism. For proving these facts we refer the reader to [16]. It is also shown there that the crossed product possesses the following *universal property*.

Let $\mathcal{B}$ be a unital C^*-algebra, $j_\mathcal{A} : \mathcal{A} \longrightarrow \mathcal{B}$ be a unital $*$-homomorphism of C^*-algebras, $j_G : G \longrightarrow U(\mathcal{B})$ be a group homomorphism from the group G to the group $U(\mathcal{B})$ of unitary elements in $\mathcal{B}$ such that

$$j_\mathcal{A}(\alpha_g(a)) = j_G(g)j_\mathcal{A}(a)j_G(g)^*$$

for all $a \in \mathcal{A}$ and $g \in G$. Thus, the pair $(j_\mathcal{A}, j_G)$ is a covariant representation of the dynamical system $(\mathcal{A}, G, \alpha)$ into the C^*-algebra $\mathcal{B}$. Then, there exists a unique unital $*$-homomorphism $\psi : \mathcal{A} \rtimes_\alpha G \longrightarrow \mathcal{B}$ satisfying the conditions

$$\psi \circ i_\mathcal{A} = j_\mathcal{A} \quad \text{and} \quad \psi \circ i_G = j_G.$$

Now let us turn to the reduced semigroup C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$. For its generating elements we introduce the notation $U_m := T_{(m,1)}$, $m \in \mathbb{Z}$, and $V_k := T_{(0,k)}$, $k \in \mathbb{Z}^\times$. It is easily seen that $U_m^* = U_{-m}$. Hence, the operator U_m is unitary for every $m \in \mathbb{Z}$.

Lemma 1. *The following properties are fulfilled.*

1) *The operator V_{-1} is self-adjoint and unitary, i.e., $V_{-1} = V_{-1}^*$, $V_{-1}^2 = I$.*

2) *For all $m \in \mathbb{Z}$, $n \in \mathbb{N}$, one has*
$$U_m V_n = V_n U_m, \quad U_m V_{-1} = V_{-1} U_{-m}, \quad V_n V_{-1} = V_{-1} V_n,$$
$$V_n^* V_{-1} = V_{-1} V_n^*, \quad U_m V_n^* = V_n^* U_m.$$

3) *The C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ is generated by two unitary operators U_1, V_{-1} and the isometries V_n, $n \in \mathbb{N}$.*

Proof. 1) The second relation is easily verified. Namely, one has $V_{-1}^2 = T_{(0,-1)}^2 = T_{(0,1)} = I$. Multiplying both sides of this equality on the left by V_{-1}^*, we get $V_{-1} = V_{-1}^*$, as required.

2) The first three relations follow from the multiplication rule of the elements in the semigroup $Z \rtimes_\varphi Z^\times$ defined by (1). Let us check one of them. For instance, we have

$$U_n V_{-1} = T_{(n,1)} T_{(0,-1)} = T_{(n,-1)} = T_{(0,-1)} T_{(-n,1)} = V_{-1} U_{-n}.$$

Further, we note that $V_n V_{-1} = V_{-1} V_n$ and $U_m V_n = V_n U_m$. Applying the involution to both these equalities and using the properties of the operators V_{-1}^* and U_n^*, we get the rest two relations.

3) If $n > 0$, then we have

$$T_{(m,n)} = U_m V_n = U_1^m V_n. \tag{4}$$

If $n < 0$, then

$$V_n = T_{(0,-n)} T_{(0,-1)} = V_{|n|} V_{-1}. \tag{5}$$

On substituting (5) into (4), we get $T_{(m,n)} = U_1^m V_{|n|} V_{-1}$. As a consequence, the reduced semigroup C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ is generated by operators U_1, V_{-1} and V_n, $n \in \mathbb{N}$. □

As is well known, the infinite dihedral group $D_\infty = \mathbb{Z} \rtimes_\psi \mathbb{Z}_2$ can be defined by means of generators and relations as follows

$$D_\infty \cong \{x, y | y^2 = 1, (xy)^2 = 1\}.$$

Of course, this isomorphism takes the elements $(1,0)$ and $(0,1)$ to the generators x and y respectively.

In the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ we consider the C^*-subalgebra $\mathfrak{B}$ generated by the elements V_n, $n \in \mathbb{N}$. Let

$$\mathrm{tr} : D_\infty \longrightarrow \mathrm{Aut}\mathfrak{B} : d \longmapsto \mathrm{id}$$

be the trivial homomorphism which takes each element $d \in D_\infty$ to the identity automorphism of $\mathfrak{B}$. Thus, we are given the dynamical system $(\mathfrak{B}, D_\infty, \mathrm{tr})$ as well as the corresponding crossed product $(\mathfrak{B} \rtimes_{\mathrm{tr}} D_\infty, i_{\mathfrak{B}}, i_{D_\infty})$.

Theorem 2. *There exists a covariant representation of the dynamical system* $(\mathfrak{B}, D_\infty, tr)$ *into the* C^**-algebra* $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$.

Proof. Firstly, let us consider the natural embedding of the C^*-subalgebra $\mathfrak{B}$ into $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ denoted by

$$\iota : \mathfrak{B} \longrightarrow C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times) : B \longmapsto B.$$

Secondly, we define a homomorphism u from the group D_∞ to the group of unitary elements of the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ in the following way. Put

$$u(x) = U_1, \ u(y) = V_{-1}. \tag{6}$$

By Lemma 1, we have the relations

$$(u(y))^2 = V_{-1}^2 = I, \quad (u(x)u(y))^2 = (U_1 V_{-1})^2 = U_1 V_{-1} U_1 V_{-1} = V_{-1} U_1^* U_1 V_{-1} = I$$

which imply that the assignment (6) defines the homomorphism between the groups.

It remains to show that the pair (ι, u) is a covariant representation of the dynamical system $(\mathfrak{B}, D_\infty, \mathrm{tr})$ into the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$, that is, $\iota(B) = u(d)\iota(B)u(d)^*$ whenever $B \in \mathfrak{B}$ and $d \in D_\infty$. To prove this relation, it suffices to show that every element of the algebra $\mathfrak{B}$ commutes with the elements U_1 and V_{-1}. But, by item 2) in Lemma 1, it is valid for the generators V_n and V_n^* of the C^*-algebra $\mathfrak{B}$ whenever $n \in \mathbb{N}$. The rest is clear. □

Corollary 2. *There exists a unique unital* $*$*-homomorphism of* C^**-algebras*

$$\psi : \mathfrak{B} \rtimes_{\mathrm{tr}} D_\infty \longrightarrow C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$$

such that $\psi(i_{\mathfrak{B}}(V_n)) = V_n$, $\psi(i_{D_\infty}(x)) = U_1$ *and* $\psi(i_{D_\infty}(y)) = V_{-1}$.

By item 3) in Lemma 1, the homomorphism ψ is surjective. Furthermore, it follows from the following theorem that ψ is an isomorphism of C^*-algebras $\mathfrak{B} \rtimes_{\mathrm{tr}} D_\infty$ and $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$.

Theorem 3. *There exists an isomorphism of C^*-algebras $\phi : C^*_r(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times) \longrightarrow \mathfrak{B} \rtimes_{\mathrm{tr}} D_\infty$ such that $\phi(U_1) = i_{D_\infty}(x)$, $\phi(V_{-1}) = i_{D_\infty}(y)$, $\phi(V_n) = i_{\mathfrak{B}}(V_n)$.*

Proof. We first take the Cartesian product $\mathbb{N} \times D_\infty$ for the multiplicative semigroup of the natural numbers $\mathbb{N}$ and the dihedral group D_∞. Here, we treat $\mathbb{N} \times D_\infty$ as the semigroup with the coordinatewise binary operation.

Next, let us define the mapping

$$\alpha : \mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times \longrightarrow \mathbb{N} \times D_\infty : (m,n) \longmapsto \begin{cases} (n,(m,0)), & \text{if } n > 0; \\ (-n,(m,1)), & \text{if } n < 0\,; \end{cases}$$

whenever $m \in \mathbb{Z}$, $n \in \mathbb{Z}^\times$. It is straightforward to verify that α is an isomorphism of the semigroups. Therefore, we have the isomorphism of C^*-algebras

$$\beta : C^*_r(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times) \longrightarrow C^*_r(\mathbb{N} \times D_\infty) : T_{(m,n)} \longmapsto T_{\alpha(m,n)}.$$

Using the unitary operator between the Hilbert spaces

$$l_2(\mathbb{N} \times D_\infty) \longrightarrow l_2(\mathbb{N}) \dot{\otimes} l_2(D_\infty) : e_{(n,d)} \longmapsto e_n \otimes e_d,$$

one gets the isomorphism of C^*-algebras given by

$$C^*_r(\mathbb{N} \times D_\infty) \longrightarrow C^*_r(\mathbb{N}) \otimes_{min} C^*_r(D_\infty) : T_{(n,d)} \longmapsto T_n \otimes S_d \tag{7}$$

whenever $n \in \mathbb{N}$ and $d \in D_\infty$ [17, Lemma 2.16].

We claim that the C^*-algebra $C^*_r(\mathbb{N})$ is isomorphic to the C^*-subalgebra $\mathfrak{B}$ of $C^*_r(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ which is generated by the operators V_n, where $n \in \mathbb{N}$. Indeed, to show this, we first consider the isometric $*$-homomorphism from [10, Theorem 2.1] given by

$$\gamma : C^*_r(\mathbb{N}) \longrightarrow C^*_r(\mathbb{N} \times D_\infty) : T_n \longmapsto T_{(n,(0,0))}.$$

Of course, the corestriction $\widetilde{\gamma} : C^*_r(\mathbb{N}) \longrightarrow \gamma(C^*_r(\mathbb{N}))$ of γ to the image set $\gamma(C^*_r(\mathbb{N}))$ is an isomorphism of C^*-algebras. Then, we consider the isomorphism of C^*-algebras

$$\beta_0 : \mathfrak{B} \longrightarrow \beta(\mathfrak{B}) : V_n \longmapsto \beta(V_n) = T_{\alpha(0,n)} = T_{(n,(0,0))}.$$

Taking the composition $\beta_0^{-1} \circ \widetilde{\gamma} : C^*_r(\mathbb{N}) \longrightarrow \mathfrak{B}$, we obtain the isomorphism between the C^*-algebras $C^*_r(\mathbb{N})$ and $\mathfrak{B}$, as claimed. Note that one has

$$\beta_0^{-1} \circ \widetilde{\gamma}(T_n) = V_n.$$

Denoting by $id : C^*_r(D_\infty) \longrightarrow C^*_r(D_\infty)$ the identity mapping and using ([18], Proposition B.13), we have the isomorphism of C^*-algebras

$$(\beta_0^{-1} \circ \widetilde{\gamma}) \otimes id : C^*_r(\mathbb{N}) \otimes_{min} C^*_r(D_\infty) \longrightarrow \mathfrak{B} \otimes_{min} C^*_r(D_\infty) \tag{8}$$

such that for all $n \in \mathbb{N}$ and $d \in D_\infty$

$$(\beta_0^{-1} \circ \widetilde{\gamma}) \otimes id(T_n \otimes S_d) = V_n \otimes S_d.$$

Further, we note that the infinite dihedral group D_∞ is amenable (see, for example, [19], Section 1). This implies that the C^*-algebra $C^*_r(D_\infty)$ is nuclear [20], and it coincides with the full C^*-algebra $C^*(D_\infty)$ ([13], Remark IV.7.68). Thus, we have

$$\mathfrak{B} \otimes_{min} C^*_r(D_\infty) = \mathfrak{B} \otimes_{max} C^*(D_\infty). \tag{9}$$

Taking the composition of isomorphisms (7) and (8) and making use of the equality (9), we get the isomorphism of C^*-algebras

$$\delta : C^*_r(\mathbb{N} \times D_\infty) \longrightarrow \mathfrak{B} \otimes_{max} C^*(D_\infty)$$

such that $\delta(T_{(n,d)}) = V_n \otimes S_d$ whenever $n \in \mathbb{N}$ and $d \in D_\infty$.

It is easy to see that the isomorphism of C^*-algebras

$$\delta \circ \beta : C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times) \longrightarrow \mathfrak{B} \otimes_{max} C^*(D_\infty)$$

takes the generating operators V_n, U_1 and V_{-1} to the elements $V_n \otimes I$, $I \otimes S_x$ and $I \otimes S_y$ respectively.

Since the group homomorphism $\mathrm{tr} : D_\infty \longrightarrow \mathrm{Aut}\mathfrak{B}$ is trivial, we have the isomorphism of C^*-algebras

$$\epsilon : \mathfrak{B} \otimes_{max} C^*(D_\infty) \longrightarrow \mathfrak{B} \rtimes_{\mathrm{tr}} D_\infty$$

such that $\epsilon(B \otimes S_d) = i_{\mathfrak{B}}(B) i_{D_\infty}(d)$ whenever $B \in \mathfrak{B}$ and $d \in D_\infty$ (see the proof of Lemma 2.73 in [14]).

Finally, we put $\phi := \epsilon \circ \delta \circ \beta$. Then, the mapping

$$\phi : C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times) \longrightarrow \mathfrak{B} \rtimes_{\mathrm{tr}} D_\infty$$

is an isomorphism of C^*-algebras such that

$$\phi(V_n) = i_{\mathfrak{B}}(V_n), \quad \phi(U_1) = i_{D_\infty}(x), \quad \phi(V_{-1}) = i_{D_\infty}(y)$$

for the generating elements V_n, U_1 and V_{-1} of the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$, as required.

The proof of the theorem is complete. □

Theorem 3 and Corollary 2 imply

Corollary 3. *The following properties are fulfilled.*

1) $\psi : \mathfrak{B} \rtimes_{\mathrm{tr}} D_\infty \longrightarrow C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ *is an isomorphism of C^*-algebras which is inverse for* $\phi : C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times) \longrightarrow \mathfrak{B} \rtimes_{\mathrm{tr}} D_\infty$;

2) ϕ *is the unique isomorphism from* $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ *onto* $\mathfrak{B} \rtimes_{\mathrm{tr}} D_\infty$ *which takes* U_1, V_{-1} *and* V_n, *to* $i_{D_\infty}(x)$, $i_{D_\infty}(y)$ *and* $i_{\mathfrak{B}}(V_n)$ *respectively.*

FUNDING

This article has been supported by the Kazan Federal University Strategic Academic Leadership Program ("PRIORITY-2030") and by the Development Program of Volga Region Mathematical Center (agreement no. 075-02-2023-944).

REFERENCES

1. L. A. Coburn, "The C^*-algebra generated by an isometry," Bull. Am. Math. Soc. **73**, 722–726 (1967).
2. L. A. Coburn, "The C^*-algebra generated by an isometry. II," Trans. Am. Math. Soc. **137**, 211–217 (1969).
3. R. G. Douglas, "On the C^*-algebra of a one-parameter semigroup of isometries," Acta Math. **128**, 143–152 (1972).
4. G. J. Murphy, "Ordered groups and Toeplitz algebras," J. Oper. Theory **18**, 303–326 (1987).
5. G. J. Murphy, "Toeplitz operators and algebras," Math. Z. **208**, 355–362 (1991).
6. X. Li, "Semigroup C^*-algebras," in *Operator Algebras and Applications, The Abel Symposium 2015,* Ed. by T. M. Carlsen, N. S. Larsen, S. Neshveyev, and C. Skau (Springer, Cham, 2016), Vol. 12, pp. 191–202.
7. E. V. Lipacheva, "Extensions of semigroups by the dihedral groups and semigroup C^*-algebras," J. Algebra Appl. (2022). https://doi.org/10.1142/S0219498824500221
8. E. V. Lipacheva, "On the representation of a semigroup C^*-algebra as a crossed product," Russian Math. **66** (8), 71–75 (2022).
9. S. A. Grigoryan, R. N. Gumerov, and E. V. Lipacheva, "On extensions of semigroups and their applications to Toeplitz algebras," Lobachevskii J. Math. **40**, 2052–2061 (2019).
10. E. V. Lipacheva, "Trivial extensions of semigroups and semigroup C^*-algebras," Ufa Math. J. **14** (2), 67–77 (2022).
11. R. N. Gumerov and E. V. Lipacheva, "Topological grading of semigroup C^*-algebras," Herald Bauman Mosc. Tech. Univ., Ser. Nat. Sci. **90** (3), 44–55 (2020).

12. E. V. Lipacheva, “On graded semigroup C^*-algebras and Hilbert modules,” Proc. Steklov Inst. Math. **313**, 120–130 (2021).
13. A. Ya. Helemskii, *Banach and Locally Convex Algebras, Oxford Science Publ.* (Clarendon, Oxford Univ. Press, New York, 1993).
14. D. P. Williams, *Crossed Products of C^*-Algebras,* Vol. 134 of *Mathematical Surveys and Monographs* (Am. Math. Soc., Providence, RI, 2007).
15. R. Exel, *Partial Dynamical Systems, Fell Bundles and Applications,* Vol. 224 of *Mathematical Surveys and Monographs* (Am. Math. Soc., Providence, RI, 2017).
16. I. Raeburn, “On crossed products and Takai duality,” Proc. Edinburgh Math. Soc. **31**, 321–330 (1988).
17. X. Li, “Semigroup C^*-algebras and amenability of semigroups,” J. Funct. Anal. **262**, 4302–4340 (2012).
18. I. Raeburn and D. P. Williams, *Morita Equivalence and Continuous-Trace C^*-Algebras,* Vol. 60 of *Mathematical Surveys and Monographs* (Am. Math. Soc., Providence, RI, 1998).
19. R. Grigorchuk and R. Yang, “Joint spectrum and the infinite dihedral group,” Proc. Steklov Inst. Math. **297**, 145–178 (2017).
20. C. Lance, “On nuclear C^*-algebras,” J. Funct. Anal. **12**, 157–176 (1973).