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Tripotents in Algebras: Ideals and Commutators

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Abstract—We establish some new properties of n-potent elements in unital algebras. Particular attention is paid to ideals in these algebras. As a consequence, we obtain the compactness conditions for the product AB of a Hilbert space tripotents A and B. In year 2011 we studied the following question: under what conditions do tripotents A and B commute? Here we try to find out when do tripotents A and B anticommute. We also determine under what conditions $A+B$ is an idempotent. We establish similarity of certain idempotents in unital algebras.

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1. INTRODUCTION

Let $\mathcal{A}$ be an algebra, $n \in \mathbb{N}$. An element $A \in \mathcal{A}$ is said to be an *n-potent* if $A^n = A$. For $n = 2$ and $n = 3$ we have the standard definitions of idempotents and tripotents, resp. Let P, Q be idempotents on a Hilbert space $\mathcal{H}$, i.e., $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{id}}$. Various properties (invertibility, Fredholm property, trace class property, positivity etc.) of the difference $X = P - Q$ have been actively studied in recent decades, see [1, 7, 9, 12, 15–17, 22–28] and references therein. If X is a trace class operator, the traces of all odd degrees of X coincide

$$\mathrm{tr}(P-Q) = \mathrm{tr}((P-Q)^{2n+1}) = \dim\ker(X-I) - \dim\ker(X+I) \in \mathbb{Z}, \tag{1}$$

here I is the identity operator on $\mathcal{H}$. If X is a compact operator, the right-hand side of (1) gives a natural "regularization" for the trace, showing that it is always an integer [2, 22]. Pairs of idempotents play an important part in the Quantum Hall Effect [3]. For idempotents P, Q, R with trace class differences $P - Q$ and $Q - R$, the equality $\mathrm{tr}(P-Q) = \mathrm{tr}(P-R) + \mathrm{tr}(R-Q)$ together with (1) imply that

$$\mathrm{tr}((P-Q)^3) = \mathrm{tr}((P-R)^3) + \mathrm{tr}((R-Q)^3). \tag{2}$$

Physical sense of additivity in (2) comes from interpretation of $\mathrm{tr}((P-Q)^3)$ as the Hall conductance. Additivity of (cubic) equation in (2) can be seen as a variant of the Ohm's law on additivity of conductance [20]. In [11, Theorem 1], a C^*-analogue of the Quantum Hall Effect is obtained and it is proved there that the trace of the differences of a wide class of symmetries from a C^*-algebra is real [11, Corollaries 2 and 3]. Any tripotent A in an algebra $\mathcal{A}$ is a difference $P - Q$ of some idempotents $P, Q \in \mathcal{A}$ with $PQ = QP = 0$ [5, Proposition 1]. Hence tripotents inherit some of the properties of idempotents [6, 13].

In this article, we establish some new properties of n-potent elements in unital algebras (Theorems 1, 2, 3). Particular attention is paid to ideals in such algebras (Theorems 4, 5). As a consequence, we obtain a compactness conditions for the product AB of a Hilbert space tripotents A and B (Corollary 1). In [5, Proposition 2] we studied the following question: under what conditions do tripotents A and B

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commute? In Theorem 6 we try to find out when do tripotents A and B anticommute. We also determine under what conditions $A+B$ is an idempotent (Theorem 7; cf. [5, p. 2157]). Let $\mathcal{A}$ be a unital algebra, let $A, B \in \mathcal{A}$ be such that $ABA = \lambda A$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. If A is an n-potent for some $n \geq 3$ then the idempotents A^{n-1}, $\lambda^{-1}AB$ and $\lambda^{-1}BA$ are pairwise similar (Theorem 8).

2. DEFINITIONS AND NOTATION

Let $\mathcal{A}$ be an algebra, $\mathcal{A}^{\text{id}} = \{A \in \mathcal{A} : A^2 = A\}$ and $\mathcal{A}^{\text{tri}} = \{A \in \mathcal{A} : A^3 = A\}$ be the set of all idempotents and all tripotents in $\mathcal{A}$, resp. For $A, B \in \mathcal{A}$ we write $A \sim B$ if there are $X, Y \in \mathcal{A}$ with $XY = A$, $YX = B$. An element $X \in \mathcal{A}$ is a *commutator*, if $X = [A, B] = AB - BA$ for some $A, B \in \mathcal{A}$. Elements $X, Y \in \mathcal{A}$ *anticommute*, if $XY = -YX$. If I is the unit of the algebra $\mathcal{A}$ and $P \in \mathcal{A}^{\text{id}}$ then $P^{\perp} = I - P \in \mathcal{A}^{\text{id}}$ and $S_P = 2P - I$ is a symmetry, i.e., $S_P^2 = I$. If $A, B \in \mathcal{A}$ are similar then $A \sim B$.

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$, $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all linear bounded operators on $\mathcal{H}$. Let $\mathcal{B}(\mathcal{H})^+$ be the positive cone in $\mathcal{B}(\mathcal{H})$, let $\mathfrak{S}_1(\mathcal{H})$ be the set of all trace class operators on $\mathcal{H}$. If $A \in \mathcal{B}(\mathcal{H})$ then $|A| = \sqrt{A^*A} \in \mathcal{B}(\mathcal{H})^+$. An operator $A \in \mathcal{B}(\mathcal{H})$ is *hyponormal*, if $A^*A \geq AA^*$; *normal*, if $A^*A = AA^*$; is a *partial isometry*, if A is isometric on $\text{Ker}(A)^{\perp}$, that is $\|A\xi\| = \|\xi\|$ for all $\xi \in \text{Ker}(A)^{\perp}$. For $\dim \mathcal{H} = n < \infty$ the algebra $\mathcal{B}(\mathcal{H})$ can be identified with the full matrix algebra $\mathbb{M}_n(\mathbb{C})$.

3. MAIN RESULTS

Lemma 1 ([5, Proposition 1]). *Let $\mathcal{A}$ be an algebra. Then for every $A \in \mathcal{A}^{tri}$ there exist $P, Q \in \mathcal{A}^{id}$ such that $A = P - Q$ and $PQ = QP = 0$. This representation is unique.*

Lemma 2. *Let $\mathcal{A}$ be an algebra and an n-potent $A \in \mathcal{A}$, $n \geq 2$. Then*

(i) A^k is an n-potent for every $k \in \mathbb{N}$;

(ii) A^{n-1} is an idempotent for every $n \geq 3$;

(iii) $\frac{1}{n-1}\sum_{k=1}^{n-1} A^k$ is an idempotent for every $n \geq 3$.

Proof. (i) We have $(A^k)^n = A^{kn} = (A^n)^k = A^k$.

(ii) We have $(A^{n-1})^2 = A^{2n-2} = A^n \cdot A^{n-2} = AA^{n-2} = A^{n-1}$.

(iii) If $B = \frac{1}{n-1}\sum_{k=1}^{n-1} A^k$ then $A^m B = BA^m = B$ for every $m \in \mathbb{N}$. □

Theorem 1. *Consider a unital algebra $\mathcal{A}$ and an n-potent $A \in \mathcal{A}$, $n \geq 3$. If there exists a right inverse element $A_r^{-1} \in \mathcal{A}$ (resp., a left inverse element $A_l^{-1} \in \mathcal{A}$) then A is invertible with $A^{-1} = A^{n-2}$.*

Proof. For a right inverse element A_r^{-1} we have

$$I = AA_r^{-1} = A^n A_r^{-1} = A^{n-1} \cdot AA_r^{-1} = A^{n-1} = A^{n-2}A = AA^{n-2},$$

i.e., $A^{-1} = A^{n-2}$. Moreover, A^{-1} is also an n-potent: $(A^{-1})^n = A^{-n} = (A^n)^{-1} = A^{-1}$ (it also follows by item (i) of Lemma 2 with $k = n-2$). In particular, for $n = 3$ we have $A^{-1} = A$, i.e., A is a symmetry. □

Note that if an element $A \in \mathcal{A}$ is right invertible and $A_r^{-1} = A^{n-2}$ then $I = AA_r^{-1} = A^{n-1}$; therefore, $A^n = A$.

Theorem 2. *Let J be an ideal in a unital algebra $\mathcal{A}$, $A, B \in \mathcal{A}^{tri}$ and $A + B = \lambda I + K$ for some $\lambda \in \mathbb{C} \setminus \{-2, 0, 2\}$ and $K \in J$. Then $AB \in J$ and $\lambda \in \{-1, 1\}$.*

Proof. Let $A = P - Q$ and $B = R - S$ be the representations of the tripotents A, B by Lemma 1, i.e., $P, Q, R, S \in \mathcal{A}^{\text{id}}$ and $PQ = QP = RS = SR = 0$. Multiply both sides of the equality $A + B = \lambda I + K$ by the idempotent P from the left and obtain

$$P + PR - PS = \lambda P + PK. \tag{3}$$

Multiply both sides of equality (3) by the idempotent S from the right and obtain $-\lambda PS = PKS$. Since $\lambda \neq 0$, we have $PS \in J$.

Next we multiply both sides of the equality $A + B = \lambda I + K$ by the idempotent Q from the left and obtain

$$-Q + QR - QS = \lambda Q + QK. \tag{4}$$

Multiply both sides of equality (4) by the idempotent R from the right and obtain $-\lambda QR = QKR$. Since $\lambda \neq 0$, we have $QR \in J$.

Multiply both sides of equality (4) by the idempotent S from the right and obtain $-(\lambda + 2)QS = QKS$. Since $\lambda \neq -2$, we have $QS \in J$. Thus $AB = PR - PS - QR + QS \in J$.

If $P \notin J$ then by (3) we have $\lambda = 1$; if $Q \notin J$ then by (4) $\lambda = -1$. Theorem is proved. □

The condition $\lambda \in \mathbb{C} \setminus \{-2, 0, 2\}$ cannot be omitted in Theorem 2. For the following pairs of tripotents:

$$1)\ A = B = \pm I \quad (\text{i.e., } \lambda = \pm 2), \quad 2)\ A = -B = \pm I \quad (\text{i.e., } \lambda = 0)$$

their products $AB \notin J$.

Corollary 1. *Let $\mathcal{A} = \mathcal{B}(\mathcal{H})$, for a separable Hilbert space $\mathcal{H}$ and assume that $\dim \mathcal{H} = +\infty$. Consider $A, B \in \mathcal{A}^{tri}$ such that $A + B$ is a non-commutator and the operators $A + B \pm 2I$ are non-compact. Then the operator AB is compact.*

Proof. Let $\mathcal{H}$ be a separable Hilbert space, $\dim \mathcal{H} = \infty$. An operator $A \in \mathcal{B}(\mathcal{H})$ is a non-commutator if and only if $A = aI + K$ for some $a \in \mathbb{C} \setminus \{0\}$ and a compact operator $K \in \mathcal{B}(\mathcal{H})$ [18, Theorem 3], [21, Chapter 19, Problem 182]. Thus the operator AB is compact and the operator $\lambda I + AB$ is a non-commutator for every $\lambda \in \mathbb{C} \setminus \{0\}$. □

On other conditions of compactness of products AB for $A, B \in \mathcal{B}(\mathcal{H})$ see [8, 10, 14] and references therein.

Theorem 3. *Let $A \in \mathcal{B}(\mathcal{H})$ be a Hermitian n-potent operator, $n \geq 2$. Then*

(i) if n is even or $A \in \mathcal{B}(\mathcal{H})^+$ then A is a projection;

(ii) if n is odd then A is a tripotent.

Proof. (i) We have for $n = 2k$, $k \in \mathbb{N}$

$$A = A^{2k} = A^k (A^*)^k = A^k (A^k)^* \in \mathcal{B}(\mathcal{H})^+.$$

Therefore, by the Spectral Theorem, A is a projection. If $A \in \mathcal{B}(\mathcal{H})^+$ then A is a projection by the Spectral Theorem.

(ii) Let $n \in \mathbb{N}$ be odd and let $A = A_+ - A_-$ be the Jordan decomposition of the Hermitian n-potent operator $A \in \mathcal{B}(\mathcal{H})$ with $A_+ A_- = 0$, where $A_+, A_- \in \mathcal{B}(\mathcal{H})^+$. Multiply both sides of the equality $A_+ - A_- = A_+^n - A_-^n$ by the operator A_+ from the right and obtain $A_+^2 = A_+^{n+1}$. Therefore, by the Spectral Theorem, A_+ is a projection. Analogously, we can prove that A_- is also a projection. Thus $A^3 = A$. Theorem is proved. □

If a tripotent $A \in \mathcal{B}(\mathcal{H})$ is hyponormal then $A^* = A$, see [6, Theorem 2]. Consider projections $P_k \in \mathcal{B}(\mathcal{H})$ with $P_k P_j = 0$ for $k \neq j$, $k, j = 1, 2, 3$, and let $\omega_1, \omega_2, \omega_3$ be the primitive cubic roots of 1. For the normal 4-potent operator $A = \omega_1 P_1 + \omega_2 P_2 + \omega_3 P_3$ we have $A^* \neq A$.

Corollary 2. *For an operator $A \in \mathcal{B}(\mathcal{H})$ the following conditions are equivalent: (i) $|A|$ is an n-potent operator for some $n \geq 2$; (ii) $|A^*|$ is an n-potent operator for some $n \geq 2$; (iii) A is a partial isometry.*

Proof. (i)⇒(iii). By item (i) of Theorem 3 the operator $|A|$ is a projection. Therefore, by the Spectral Theorem $|A|^2 = A^* A$ is a projection and A is a partial isometry by [21, Chapter 13, Problem 98].

(iii)⇒(i). If A is a partial isometry, then $A^* A = |A|^2$ is a projection by [21, Chapter 13, Problem 98]. Hence the operator $|A| = \sqrt{|A|^2}$ is a projection by the Spectral Theorem.

(ii)⇔(iii). An operator $A \in \mathcal{B}(\mathcal{H})$ is a partial isometry if and only if A^* is a partial isometry [26, Theorem 2.3.3]. □

Theorem 4. *Let J be an ideal in an algebra $\mathcal{A}$. Let $A, B, X \in \mathcal{A}$, and let A be a k-potent, B be an n-potent for some $k, n \in \mathbb{N}$. Then the following conditions are equivalent:*

(i) $AXB \in J$;

(ii) $A^jXB^m \in J$ *for some* $1 \le j \le k$ and $1 \le m \le n$.

Proof. (i)⇒(ii). If $j = 1$ and $m > 1$ then $AXB^m = AXB \cdot B^{m-1} \in J$. If $j > 1$ and $m = 1$ then $A^jXB = A^{j-1} \cdot AXB \in J$. If $j, m > 1$ then $A^jXB^m = A^{j-1} \cdot AXB \cdot B^{m-1} \in J$.

(ii) ⇒ (i). If $j = k$ and $m < n$ then $AXB = AXB^m \cdot B^{n-m} \in J$. If $j < k$ and $m = n$ then $AXB = A^{k-j} \cdot A^jXB \in J$. If $j < k$ and $m < n$ then $AXB = A^{k-j} \cdot A^jXB^m \cdot B^{n-m} \in J$.

In particular, $A \in J \Leftrightarrow A^j \in J$ for some $1 \le j \le k$. □

Theorem 5. *Let J be an ideal in an algebra $\mathcal{A}$, $A, B \in \mathcal{A}^{tri}$ and $A = P_1 - Q_1$, $B = P_2 - Q_2$ be the representations of Lemma 1. Then the following conditions are equivalent:*

(i) $A - B \in J$;

(ii) $P_1 - P_2, Q_1 - Q_2 \in J$.

Proof. We have $P_k, Q_k \in \mathcal{A}^{\mathrm{id}}$ and $P_kQ_k = Q_kP_k = 0$ for $k = 1, 2$.

(i) ⇒ (ii). We apply the scheme of the proof of [9, Corollary 5]. The elements $A^2 = P_1 + Q_1$, $B^2 = P_2 + Q_2$ lie in $\mathcal{A}^{\mathrm{id}}$ by item (ii) of Lemma 2. Since $A - B = P_1 - Q_1 - P_2 + Q_2 \in J$, the element

$$A^2 - B^2 = \frac{1}{2}((A + B)(A - B) + (A - B)(A + B)) = P_1 + Q_1 - P_2 - Q_2$$

also belongs to J. Therefore, the elements

$$P_1 - P_2 = \frac{1}{2}(A - B + A^2 - B^2), \quad Q_1 - Q_2 = -\frac{1}{2}(A - B - (A^2 - B^2))$$

lie in J.

(ii) ⇒ (i). We have $A - B = P_1 - P_2 - (Q_1 - Q_2) \in J$. □

Lemma 3. *Let $\mathcal{A}$ be an algebra and $A, B \in \mathcal{A}$ be such that $AB = -BA$, i.e., A and B anticommute. Then $A^kB^{2n} = B^{2n}A^k$ and $A^{2k+1}B^{2n+1} = -B^{2n+1}A^{2k+1}$ for all $k, n \in \mathbb{N}$.*

Proof. We have $AB^2 = AB \cdot B = -BA \cdot B = -B \cdot AB = -B \cdot (-BA) = B^2A$. Therefore,

$$A^kB^2 = A^{k-1} \cdot AB^2 = A^{k-1} \cdot B^2A = A^{k-2} \cdot AB^2 \cdot A = \cdots = B^2A^k$$

for all $k \in \mathbb{N}$. Thus $A^kB^{2n} = B^{2n}A^k$ for all $k, n \in \mathbb{N}$. We have

$$\begin{aligned} A^{2k+1}B^{2n+1} &= A^{2k+1}B^{2n} \cdot B = B^{2n}A^{2k+1} \cdot B = B^{2n}A^{2k} \cdot AB = -1 \cdot B^{2n}A^{2k} \cdot BA \\ &= -1 \cdot B^{2n}A^{2k-1} \cdot AB \cdot A = (-1)^2B^{2n}A^{2k-1} \cdot BA^2 = \cdots = (-1)^{2k+1}B^{2n+1}A^{2k+1} \end{aligned}$$

for all $k, n \in \mathbb{N}$. □

Theorem 6. *Let $\mathcal{A}$ be an algebra, $A \in \mathcal{A}$ and $B \in \mathcal{A}^{tri}$, and let $B = P - Q$ be the representation of Lemma 1. Then the following conditions are equivalent:*

(i) $AB = -BA$, i.e., A and B anticommute;

(ii) $AP = QA$ and $AQ = PA$.

Proof. (i) ⇒ (ii). We have $B^2 = P + Q$,

$$A(P - Q) = -(P - Q)A, \tag{5}$$

and by Lemma 3 obtain

$$A(P + Q) = (P + Q)A. \tag{6}$$

Add term by term equalities (5) and (6) and conclude that $AP = QA$. Subtract term by term relation (6) from (5) and obtain $AQ = PA$.

(ii) ⇒ (i). We have $AB = A(P - Q) = QA - PA = -BA$. □

Let $\mathcal{A}$ be a unital algebra, $A \in \mathcal{A}^{\mathrm{tri}}$, and let $A = P - Q$ be the representation of Lemma 1. Then $B = P^{\perp} - Q \in \mathcal{A}^{\mathrm{id}}$ and $AB = BA = 0$.

Corollary 3. *Let $\mathcal{A}$ be an algebra, $A \in \mathcal{A}$ and $P \in \mathcal{A}^{id}$. Then the following conditions are equivalent: (i) $AP = -PA$; (ii) $AP = PA = 0$.*

Proof. Put $Q = 0$ in Theorem 6. □

In $\mathbb{M}_2(\mathbb{C})$ for the tripotents

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we have $AB = -BA$. Moreover, A and B are Hermitian symmetries. Let $n \in \mathbb{N}$ and let $X, Y \in \mathbb{M}_n(\mathbb{C})$ anticommute. Then $\mathrm{tr}(XY) = \mathrm{tr}(YX) = 0$ and the matrices XY, YX are commutators by [21, Chapter 19, Problem 182]. If n is odd then $\det(XY) = 0$.

Theorem 7. *Let $\mathcal{A}$ be a unital algebra, $A \in \mathcal{A}^{tri}$ and $B \in \mathcal{A}$ with $B^2 = I$. Then the following conditions are equivalent:*

(i) $A + B \in \mathcal{A}^{\mathrm{id}}$;

(ii) $Q = 0$ and $(P - R)^2 = I$, where $A = P - Q$ is the representation of Lemma 1, $R = \frac{B+I}{2} \in \mathcal{A}^{\mathrm{id}}$.

Proof. (i) $\Rightarrow$ (ii). We have $A^2 = P + Q$. If $A + B \in \mathcal{A}^{\mathrm{id}}$ then $A^2 + AB + BA + I = A + B$, i.e.,

$$2Q - P - R + PR + RP - QR - RQ + I = 0. \tag{7}$$

Multiply both sides of equality (7) by the idempotent P from the left and obtain

$$PRP - PRQ = 0. \tag{8}$$

Multiply both sides of equality (8) by the idempotent P from the right and find that $PRP = 0$. Therefore, we have $PRQ = 0$, see (8). Multiply both sides of equality (7) by the idempotent Q from the left and by the idempotent P from the right and obtain $QRP = 0$. Multiply both sides of equality (7) by the idempotent Q from the left and the right and obtain $QRQ = Q$.

Multiply both sides of equality (7) by the tripotent $P - Q$ from the left, take into account the relations

$$PRP = PRQ = QRP = 0, \quad QRQ = Q$$

and conclude that $Q = 0$. Thus, $A = P \in \mathcal{A}^{\mathrm{id}}$ and (7) turns into $(P - R)^2 = I$.

(ii) $\Rightarrow$ (i). We have equality (7). □

Corollary 4. *Let $\mathcal{A}$ be a unital algebra, $A, B \in \mathcal{A}$ with $A^2 = B^2 = I$. Then the following conditions are equivalent: (i) $A + B \in \mathcal{A}^{\mathrm{id}}$; (ii) $A = -B = I$.*

Proof. (i) $\Rightarrow$ (ii). Since $A = P \in \mathcal{A}^{\mathrm{id}}$ and $A^2 = I$, we have $A = P = I$. Since $(P - R)^2 = I$, we have $I - R = I$ and $R = 0$. Thus, $B = 2R - I = -I$. □

Theorem 8. *Let $\mathcal{A}$ be a unital algebra, let $A, B \in \mathcal{A}$ be such that $ABA = \lambda A$ for some $\lambda \in \mathbb{C} \setminus \{0\}$.*

(i) If A is an n-potent for some $n \geq 3$ then the idempotents A^{n-1}, $\lambda^{-1}AB$ and $\lambda^{-1}BA$ are pairwise similar. If $\mathcal{A}$ acts on a vector space $\mathcal{E}$, then we have $Im(A^{n-1}) = Im(\lambda^{-1}AB)$ and $\mathrm{Ker}(A^{n-1}) = \mathrm{Ker}(\lambda^{-1}BA)$.

(ii) If B is a $2n$-potent then $P = \lambda^{-1}B^nAB^n$ lies in $\mathcal{A}^{id}$ and $B^{2n-1}P = PB^{2n-1} = P$.

Proof. By [17, Lemma 3.8] the elements $P = \lambda^{-1}AB$ and $Q = \lambda^{-1}BA$ lie in $\mathcal{A}^{\mathrm{id}}$.

(i). We have $A^{n-1} \in \mathcal{A}^{\mathrm{id}}$ by item (ii) of Lemma 2 and

$$A^{n-1} \cdot \lambda^{-1}AB = \lambda^{-1}AB, \quad \lambda^{-1}AB \cdot A^{n-1} = \lambda^{-1}ABA \cdot A^{n-2} = A^{n-1}$$

(resp., $A^{n-1} \cdot \lambda^{-1}BA = \lambda^{-1}A^{n-2} \cdot ABA = A^{n-2}A = A^{n-1}$, $\lambda^{-1}BA \cdot A^{n-1} = \lambda^{-1}BA$). Then, we apply [15, Lemma 2] and conclude that A^{n-1} and $\lambda^{-1}AB$ (resp., A^{n-1} and $\lambda^{-1}BA$) are similar.

If $\mathcal{A}$ acts on a vector space $\mathcal{E}$ then by [19, Lemma 2] we have $\mathrm{Im}(A^{n-1}) = \mathrm{Im}(\lambda^{-1}AB)$ and $\mathrm{Ker}(A^{n-1}) = \mathrm{Ker}(\lambda^{-1}BA)$. Since every similarity relation is an equivalence, the idempotents $\lambda^{-1}AB$ and $\lambda^{-1}BA$ are also similar.

(ii) We have

$$P = \lambda^{-1}B^nAB^n = B^n \cdot \lambda^{-1}A \cdot B^n = B^n \cdot \lambda^{-2}ABA \cdot B^n = \lambda^{-2}B^nAB^n \cdot B^nAB^n = P^2.$$

Thus, $P \in \mathcal{A}^{\mathrm{id}}$. Since $B^{2n-1} \cdot B^n = B^n \cdot B^{2n-1} = B^{3n-1} = B^{2n}B^{n-1} = B^n$, we have $B^{2n-1}P = PB^{2n-1} = P$. Recall that $B^{2n-1} \in \mathcal{A}^{\mathrm{id}}$ by item (ii) of Lemma 2. □

Consider the following complex 2×2 matrices

$$A = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \lambda & \mu \\ 0 & \nu \end{pmatrix}.$$

Then $A \in \mathbb{M}_2(\mathbb{C})^{\mathrm{id}}$ and $ABA = \lambda A$. Recall that for an arbitrary $A \in \mathbb{M}_n(\mathbb{C})$ there exists a pseudo-inverse $B \in \mathbb{M}_n(\mathbb{C})$ such that $ABA = A$ (see [26, Theorem 1.4.15]).

If $A, B \in \mathbb{M}_n(\mathbb{C})$ and $A \sim B$ then $\det(A) = \det(B)$ and $\mathrm{tr}(A) = \mathrm{tr}(B)$. Let $\mathcal{A} = \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space, $\dim \mathcal{H} = \infty$. Then there exist operators $A \in \mathcal{A}^+$ and $B \in \mathcal{A}$ such that $A \sim B$, $A \in \mathfrak{S}_1(\mathcal{H})$, but $B \notin \mathfrak{S}_1(\mathcal{H})$. Hint: for some projections $P, Q \in \mathcal{B}(\mathcal{H})$ we have $PQP \in \mathfrak{S}_1(\mathcal{H})$, but $QP \notin \mathfrak{S}_1(\mathcal{H})$, see [4, Remark 1].

Theorem 9. *Let $\mathcal{A}$ be an algebra and let $A = P - Q$ and $B = S - T$ be the representations of a tripotents $A, B \in \mathcal{A}^{tri}$ by Lemma 1, i.e., $P, Q, S, T \in \mathcal{A}^{id}$ and $PQ = QP = ST = TS = 0$. If $A \sim B$ then $A^2 \sim B^2$, $P \sim S$ and $Q \sim T$. Conversely, if $P \sim S$ and $Q \sim T$, then $A \sim B$ and $A^2 \sim B^2$.*

Proof. Let $X, Y \in \mathcal{A}$ be such that $A = XY$ and $B = YX$. Then the elements $A^2 = P + Q$, $B^2 = S + T$ lie in $\mathcal{A}^{\mathrm{id}}$ and $A^2 = XYX \cdot Y$ and $B^2 = Y \cdot XYX$. Thus, $A^2 \sim B^2$ and we have

$$P = \frac{A + A^2}{2} = X \cdot \frac{Y + YXY}{2} \quad \text{and} \quad S = \frac{B + B^2}{2} = \frac{Y + YXY}{2} \cdot X,$$

$$Q = \frac{A^2 - A}{2} = X \cdot \frac{YXY - Y}{2} \quad \text{and} \quad T = \frac{B^2 - B}{2} = \frac{YXY - Y}{2} \cdot X,$$

i.e., $P \sim S$ and $Q \sim T$.

Assume now that $P \sim S$ and $Q \sim T$, i.e., $P = EF$, $S = FE$ and $Q = UV$, $T = VU$ for some $E, F, U, V \in \mathcal{A}$. Then

$$EFUV = UVEF = FEVU = VUFE = 0$$

and we have

$$A = EF - UV = (EFE - UVU)(FEF + VUV),$$
$$B = FE - VU = (FEF + VUV)(EFE - UVU);$$

$$A^2 = EF + UV = (EFE + UVU)(FEF + VUV),$$
$$B^2 = FE + VU = (FEF + VUV)(EFE + UVU).$$

Thus, $A \sim B$ and $A^2 \sim B^2$. Theorem is proved. □

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REFERENCES

1. E. Andruchow, "Operators which are the difference of two projections," J. Math. Anal. Appl. **420**, 1634–1653 (2014).
2. J. Avron, R. Seiler, and B. Simon, "The index of a pair of projections," J. Funct. Anal. **120**, 220–237 (1994).
3. J. Bellissard, A. van Elst, and H. Schulz-Baldes, "The noncommutative geometry of the quantum Hall effect," J. Math. Phys. **35**, 5373–5451 (1994).
4. A. Bikchentaev, "Majorization for products of measurable operators," Int. J. Theor. Phys. **37**, 571–576 (1998).
5. A. M. Bikchentaev and R. S. Yakushev, "Representation of tripotents and representations via tripotents," Linear Algebra Appl. **435**, 2156–2165 (2011).
6. A. M. Bikchentaev, "Tripotents in algebras: Invertibility and hyponormality," Lobachevskii J. Math. **35**, 281–285 (2014).

7. A. M. Bikchentaev, “On idempotent τ-measurable operators affiliated to a von Neumann algebra,” Math. Notes **100**, 515–525 (2016).
8. A. Bikchentaev, “Integrable products of measurable operators,” Lobachevskii J. Math. **37**, 397–403 (2016).
9. A. M. Bikchentaev, “Differences of idempotents in C^*-algebras,” Sib. Math. J. **58**, 183–189 (2017).
10. A. M. Bikchentaev, “On τ-compactness of products of τ-measurable operators,” Int. J. Theor. Phys. **56**, 3819–3830 (2017).
11. A. M. Bikchentaev, “Differences of idempotents in C^*-algebras and the quantum Hall effect,” Theor. Math. Phys. **195**, 557–562 (2018).
12. A. M. Bikchentaev, “Trace and differences of idempotents in C^*-algebras,” Math. Notes **105**, 641–648 (2019).
13. A. M. Bikchentaev, “Rearrangements of tripotents and differences of isometries in semifinite von Neumann algebras,” Lobachevskii J. Math. **40**, 1450–1454 (2019).
14. A. M. Bikchentaev, “On the τ-compactness of the product of τ-measurable operators adjoint to a semifinite von Neumann algebra,” J. Math. Sci. **241**, 458–468 (2019).
15. A. M. Bikchentaev and Kh. Fawwaz, “Differences and commutators of idempotents in C^*-algebras,” Russ. Math. **65** (8), 13–22 (2021).
16. A. M. Bikchentaev and A. N. Sherstnev, “Studies on noncommutative measure theory in Kazan University (1968–2018),” Int. J. Theor. Phys. **60**, 585–596 (2021).
17. A. Bikchentaev, “Differences and commutators of projections on a Hilbert space,” Int. J. Theor. Phys. **61**, 2 (2022).
18. A. Brown and C. Pearcy, “Structure of commutators of operators,” Ann. Math. 2 **82**, 112–127 (1965).
19. G. Chevalier, “Automorphisms of an orthomodular poset of projections,” Int. J. Theor. Phys. **44**, 985–998 (2005).
20. F. Gesztesy, “From mathematical physics to analysis: A walk in Barry Simon’s mathematical garden, II,” Not. Am. Math. Soc. **63**, 878–889 (2016).
21. P. R. Halmos, *A Hilbert Space Problem Book,* 2nd ed., Vol. 17 of *Encyclopedia of Mathematics and its Applications,* Vol. 19 of *Graduate Texts in Mathematics* (Springer, New York, 1982).
22. N. J. Kalton, “A note on pairs of projections,” Bol. Soc. Mat. Mex. 3 **3**, 309–311 (1997).
23. J. J. Koliha and V. Rakočević, “Invertibility of the difference of idempotents,” Lin. Multilin. Algebra **51**, 97–110 (2003).
24. J. J. Koliha, V. Rakočević, and I. Straškraba, “The difference and sum of projectors,” Linear Algebra Appl. **388**, 279–288 (2004).
25. J. J. Koliha and V. Rakočević, “Fredholm properties of the difference of orthogonal projections in a Hilbert space,” Integral Equat. Oper. Theory **52**, 125–134 (2005).
26. G. J. Murphy, *C^*-algebras and Operator Theory* (Academic, Boston, MA, 1990).
27. W. Shi, G. Ji, and H. Du, “Pairs of orthogonal projections with a fixed difference,” Linear Algebra Appl. **489**, 288–297 (2016).
28. K. Zuo, “Nonsingularity of the difference and the sum of two idempotent matrices,” Linear Algebra Appl. **433**, 476–482 (2010).