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Characterization of Tracial Functionals on Von Neumann Algebras

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Abstract—It is proved that the inequality

$$\varphi(A) \le \varphi(|A + \mathrm{i}B|) \quad \text{for all} \quad A \in \mathcal{A}^+ \quad \text{and} \quad B \in \mathcal{A}^{\mathrm{sa}}$$

characterizes tracial functionals among all positive normal functionals φ on a von Neumann algebra $\mathcal{A}$. This strengthens the L. T. Gardner's characterization (1979). As a consequence, a criterion for commutativity of von Neumann algebras is obtained. Also we give a characterization of traces in a wide class of weights on a von Neumann algebra via this inequality. Every faithful normal semifinite trace φ on a von Neumann algebra $\mathcal{A}$ satisfies this relation. Let $|||\cdot|||$ be a unitarily invariant norm on a unital C^*-algebra $\mathcal{A}$. Then $|||A||| \le |||A + \mathrm{i}B|||$ for all $A \in \mathcal{A}^+$ and $B \in \mathcal{A}^{\mathrm{sa}}$.

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1. INTRODUCTION

Dimension functions and traces on C^*-algebras are fundamental tools in the operator theory and its applications. Hence the problem of characterization of traces in a wide classes of weights on C^*-algebras is important and attracts the attention of a large group of mathematicians, see [1–11].

If φ is a tracial positive normal linear functional on a von Neumann algebra $\mathcal{A}$, and p, q are positive numbers such that $1/p + 1/q = 1$, then the following hold:

- Hölder's inequality [12, Chapter IX, Theorem 2.13], [11, Theorem 5]:

$$\varphi(|XY|) \le \varphi(X^p)^{1/p}\varphi(Y^q)^{1/q} \quad \text{for all} \quad X, Y \in \mathcal{A}^+;$$

- Cauchy–Schwarz–Buniakowski inequality [13, Theorem 4.21]:

$$\varphi(|XY|^{1/2}) \le \varphi(X)^{1/2}\varphi(Y)^{1/2} \quad \text{for all} \quad X, Y \in \mathcal{A}^+;$$

- Golden–Thompson inequality [14, Theorem 4]:

$$\varphi(e^{X+Y}) \le \varphi(e^{X/2}e^{Y}e^{X/2}) \quad \text{for all} \quad X, Y \in \mathcal{A}^{\mathrm{sa}};$$

- Peierls–Bogoliubov inequality [14, Theorem 7]:

$$\varphi(e^X)\exp\frac{\varphi(e^{X/2}Ye^{X/2})}{\varphi(e^X)} \le \varphi(e^{X+Y}) \quad \text{for all} \quad X, Y \in \mathcal{A}^+.$$

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H. Araki in [15] has proved the following inequality:

$$\mathrm{tr}((X^{1/2}YX^{1/2})^{rp}) \le \mathrm{tr}((X^{r/2}Y^rX^{r/2})^p), \quad r \ge 1, \quad p > 0.$$

Here X, Y are positive operators on a Hilbert space $\mathcal{H}$. This inequality generalizes the Lieb and Thirring inequalities, and is close in spirit to the Golden—Thompson inequality (see [16, § 8]).

In general any natural trace inequality is sharp in the sense that the trace is the only (positive linear) functional that satisfies the inequality. It is well-known that either of inequalities: Hölder, Cauchy—Schwarz—Buniakowski, Golden—Tompson, Peierls—Bogoliubov, Araki—Lieb—Thirring etc., limited only to projections characterizes the tracial functionals among all positive normal functionals φ on a von Neumann algebra $\mathcal{A}$, see [17—23].

In this paper, we characterize the trace via the well-known inequality

$$\varphi(A) \le \varphi(|A + \mathrm{i}B|) \quad \text{for all} \quad A \in \mathcal{A}^+ \quad \text{and} \quad B \in \mathcal{A}^{\mathrm{sa}}, \tag{1}$$

where $\mathrm{i} \in \mathbb{C}$ with $\mathrm{i}^2 = -1$ (Theorem 1). It is shown that this strengthens the Gardner's characterization [8] (Corollary 2). As a consequence, a criterion for commutativity of von Neumann algebras is obtained (Corollary 3). Also we give a characterization of traces in a wide class of weights on a von Neumann algebra via inequality (1) (Corollary 4). Every faithful normal semifinite trace φ on a von Neumann algebra $\mathcal{A}$ satisfies relation (1) (Theorem 2). Let $|||\cdot|||$ be a unitarily invariant norm on a unital C^*-algebra $\mathcal{A}$. Then $|||A||| \le |||A + \mathrm{i}B|||$ for all $A \in \mathcal{A}^+$ and $B \in \mathcal{A}^{\mathrm{sa}}$ (Theorem 3). Recall one interesting fact in the finite-dimensional case:

If the matrix $A \in \mathbb{M}_n(\mathbb{C})$ in the Cartesian decomposition $T = A + \mathrm{i}B$ is positive then for determinants we have $|\det T| \ge \det A$, see [24, Corollary VI.7.5].

2. DEFINITIONS AND NOTATION

A C^*-algebra is a complex Banach $*$-algebra $\mathcal{A}$ such that $||A^*A|| = ||A||^2$ for all $A \in \mathcal{A}$. For a C^*-algebra $\mathcal{A}$ by $\mathcal{A}^{\mathrm{pr}}$, $\mathcal{A}^{\mathrm{sa}}$ and $\mathcal{A}^+$ we denote its subsets of projections ($A = A^* = A^2$), Hermitian elements ($A^* = A$) and positive elements, respectively. If $A \in \mathcal{A}$, then $|A| = \sqrt{A^*A} \in \mathcal{A}^+$. If I is the unit of the algebra $\mathcal{A}$ and $P \in \mathcal{A}^{\mathrm{pr}}$ then $P^\perp = I - P$.

A mapping $\varphi : \mathcal{A}^+ \to [0, +\infty]$ is called *a weight* on a C^*-algebra $\mathcal{A}$, if $\varphi(X + Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{A}^+$, $\lambda \ge 0$ (moreover, $0 \cdot (+\infty) \equiv 0$). For a weight φ define

$$\mathfrak{M}_\varphi^+ = \{X \in \mathcal{A}^+ : \ \varphi(X) < +\infty\}, \quad \mathfrak{M}_\varphi = \mathrm{lin}_{\mathbb{C}}\mathfrak{M}_\varphi^+.$$

The restriction $\varphi|_{\mathfrak{M}_\varphi^+}$ can always be extended by linearity to a functional on $\mathfrak{M}_\varphi$, which we denote by the same letter φ. Such an extension allows us to identify finite weights (i.e., $\varphi(X) < +\infty$ for all $X \in \mathcal{A}^+$) with positive functionals on $\mathcal{A}$. A positive linear functional φ on $\mathcal{A}$ with $||\varphi|| = 1$ is called *a state*. A weight φ is called *faithful*, if $\varphi(X) = 0$ $(X \in \mathcal{A}^+) \Rightarrow X = 0$; *a trace*, if $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{A}$. A trace φ on a C^*-algebra $\mathcal{A}$ is called *semifinite*, if $\varphi(A) = \sup\{\varphi(B) : \ B \in \mathcal{A}^+, \ B \le A, \ \varphi(B) < +\infty\}$ for every $A \in \mathcal{A}^+$.

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$, $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all linear bounded operators on $\mathcal{H}$. By Gelfand—Naimark theorem every C^*-algebra is isometrically isomorphic to a concrete C^*-algebra of operators on a Hilbert space $\mathcal{H}$ [25, II.6.4.10]. By the commutant of a set $\mathcal{X} \subset \mathcal{B}(\mathcal{H})$ we mean the set

$$\mathcal{X}' = \{Y \in \mathcal{B}(\mathcal{H}) : \ XY = YX \ \text{ for all } \ X \in \mathcal{X}\}.$$

A $*$-subalgebra $\mathcal{A}$ of the algebra $\mathcal{B}(\mathcal{H})$ is said to be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, if $\mathcal{A} = \mathcal{A}''$. For $P, Q \in \mathcal{A}^{\mathrm{pr}}$ we write $P \sim Q$ (the Murray—von Neumann equivalence), if $P = U^*U$ and $Q = UU^*$ for some $U \in \mathcal{A}$.

A weight φ on von Neumann algebra $\mathcal{A}$ is called *normal*, if $X_i \nearrow X$ $(X_i, X \in \mathcal{A}^+) \Longrightarrow \varphi(X) = \sup \varphi(X_i)$; *semifinite*, if the set $\mathfrak{M}_\varphi$ is ultraweakly dense in $\mathcal{A}$ (see [26, Definition VII.1.1]).

Using Upmeier's results [27], it is actually proved in [28, Theorem 1.4.2]Ayu86 that a weight on a von Neumann algebra $\mathcal{A}$ is a trace if and only if $\varphi(SAS) = \varphi(A)$ for any $A \in \mathcal{A}^+$ and a symmetry $S \in \mathcal{A}^{\mathrm{sa}}$.

3. TRACE CHARACTERIZATION ON VON NEUMANN ALGEBRAS

Let us recall Taylor's formula with Peano's remainder.

Lemma 1. *If* $b \in \mathbb{R}$ *then*

$$(1+x)^b = 1 + bx + \frac{1}{2!}b(b-1)x^2 + \ldots + \frac{1}{n!}b(b-1)\cdots(b-n+1)x^n + o(x^n) \quad as \quad x \to 0.$$

Theorem 1. *For a positive normal linear functional* φ *on a von Neumann algebra* $\mathcal{A}$ *the following conditions are equivalent:*

(i) φ *is tracial;*

(ii) $\varphi(A) \le \varphi(|A + \mathrm{i}B|)$ *for all* $A \in \mathcal{A}^+$ *and* $B \in \mathcal{A}^{sa}$*, where* $\mathrm{i} \in \mathbb{C}$ with $\mathrm{i}^2 = -1$.

Proof. (i) ⇒ (ii). The finite traces on a C^*-algebra $\mathcal{A}$ are precisely those (positive) linear functionals φ on $\mathcal{A}$ which satisfy $|\varphi(X)| \le \varphi(|X|)$ for all $X \in \mathcal{A}$, see [8]. Since $\varphi(A) \in \mathbb{R}^+$ and $\varphi(B) \in \mathbb{R}$, we have $\varphi(A) \le |\varphi(A + \mathrm{i}B)|$.

Let us show that for an arbitrary von Neumann algebra, the proof of the inverse implication (i.e., (ii) ⇒ (i)) can be reduced to the case of the algebra $\mathbb{M}_2(\mathbb{C})$ just as this was done in a number of other similar cases (see [8] or [29]).

It is well known [8] that a positive normal linear functional φ on a von Neumann algebra $\mathcal{A}$ is tracial if and only if $\varphi(P) = \varphi(Q)$ for all $P, Q \in \mathcal{A}^{\mathrm{pr}}$ with $PQ = 0$ and $P \sim Q$ (also see [29, Lemma 2]). Assume that a *-algebra $\mathcal{N}$ in the reduced algebra $(P+Q)\mathcal{A}(P+Q)$ is generated by a partial isometry $V \in \mathcal{A}$ realizing the equivalence of P and Q. Then $\mathcal{N}$ is *-isomorphic to $\mathbb{M}_2(\mathbb{C})$, while inequality (ii) remains valid for operators from $\mathcal{N}$ and the restriction of the functional $\varphi|\mathcal{N}$. We shall show that such a restriction is a tracial functional on $\mathcal{N}$; therefore, $\varphi(P) = \varphi(Q)$.

As it is well known, every linear functional φ on $\mathbb{M}_2(\mathbb{C})$ can be represented in the form $\varphi(\cdot) = \mathrm{tr}(S_\varphi \cdot)$. The two-by-two matrix S_φ is called the density matrix of φ. It is easily seen that without loss of generality we can assume that

$$S_\varphi = \mathrm{diag}\left(\frac{1}{2} - s, \frac{1}{2} + s\right), \quad 0 \le s \le \frac{1}{2}.$$

Thus $\varphi(X)$ equals $(1/2 - s)x_{11} + (1/2 + s)x_{22}$ for $X = [x_{ij}]_{i,j=1}^2$ in $\mathbb{M}_2(\mathbb{C})$.

Let $\delta \in \mathbb{C}$ with $|\delta| = 1$ and $0 \le t \le 1$. By $R^{(t,\delta)}$ we denote the projection

$$R^{(t,\delta)} = \begin{pmatrix} t & \delta\sqrt{t - t^2} \\ \bar{\delta}\sqrt{t - t^2} & 1 - t \end{pmatrix} \in \mathbb{M}_2(\mathbb{C}).$$

Let us put $A = R^{(1/2+\varepsilon,1)}$ and

$$B = \begin{pmatrix} 0 & \mathrm{i}\varepsilon \\ -\mathrm{i}\varepsilon & 0 \end{pmatrix}$$

for $0 < \varepsilon < 1/2$. Then $\varphi(A) = \mathrm{tr}(S_\varphi A) = (1/2 - s)(1/2 + \varepsilon) + (1/2 + s)(1/2 - \varepsilon) = 1/2 - 2s\varepsilon$. Put $f(\varepsilon) = \sqrt{1/4 - \varepsilon^2}$, then the matrix

$$|A + \mathrm{i}B|^2 = \begin{pmatrix} \frac{1}{2} + \varepsilon + \varepsilon^2 + 2\varepsilon f(\varepsilon) & f(\varepsilon) - 2\varepsilon^2 \\ f(\varepsilon) - 2\varepsilon^2 & \frac{1}{2} - \varepsilon + \varepsilon^2 - 2\varepsilon f(\varepsilon) \end{pmatrix}$$

has the characteristic equation $\lambda^2 - (1 + 2\varepsilon^2)\lambda + \varepsilon^4 = 0$. Therefore,

$$\lambda_1 = \frac{1 + 2\varepsilon^2 + \sqrt{1 + 4\varepsilon^2}}{2} \qquad \text{and} \quad \lambda_2 = \frac{1 + 2\varepsilon^2 - \sqrt{1 + 4\varepsilon^2}}{2}.$$

By the Taylor's formula with the reminder in the Peano form (see Lemma 1 with $b = 1/2$) we obtain

$$f(\varepsilon) = \frac{1}{2}\sqrt{1 - 4\varepsilon^2} = \frac{1}{2} - \varepsilon^2 - \varepsilon^4 + o(\varepsilon^5), \quad \sqrt{1 + 4\varepsilon^2} = 1 + 2\varepsilon^2 - 2\varepsilon^4 + o(\varepsilon^5) \quad \text{as} \quad \varepsilon \to 0+.$$

Hence

$$\lambda_1 = 1 + 2\varepsilon^2 - \varepsilon^4 + o(\varepsilon^5), \quad \lambda_2 = \varepsilon^4 + o(\varepsilon^5) \quad \text{as} \quad \varepsilon \to 0+.$$

Let us represent by the finite-dimensional Spectral Theorem

$$|A + \mathrm{i}B|^2 = \lambda_1 R^{(t,1)} + \lambda_2 R^{(t,1)\perp} = \lambda_1 R^{(t,1)} + \lambda_2 R^{(1-t,-1)}$$

and find the unknown parameter $t \in [0,1]$ from the equation

$$\frac{1}{2} + \varepsilon + \varepsilon^2 + 2\varepsilon f(\varepsilon) = \lambda_1 t + \lambda_2(1-t), \quad \text{i.e.,}$$

$$\frac{1}{2} + 2\varepsilon + \varepsilon^2 - 2\varepsilon^3 - 2\varepsilon^5 + o(\varepsilon^5) = (1 + 2\varepsilon^2 - \varepsilon^4 + o(\varepsilon^5))t + (\varepsilon^4 + o(\varepsilon^5))(1-t) \quad \text{as} \quad \varepsilon \to 0+.$$

Thus

$$t = \frac{\frac{1}{2} + 2\varepsilon + \varepsilon^2 - 2\varepsilon^3 - \varepsilon^4 - 2\varepsilon^5 + o(\varepsilon^5)}{1 + 2\varepsilon^2 - 2\varepsilon^4 + o(\varepsilon^5)} \quad \text{as} \quad \varepsilon \to 0+$$

and by Lemma 1 with $b = -1$ we have

$$t = \left(\frac{1}{2} + 2\varepsilon + \varepsilon^2 - 2\varepsilon^3 - \varepsilon^4 - 2\varepsilon^5 + o(\varepsilon^5)\right)(1 - 2\varepsilon^2 + 6\varepsilon^4 + o(\varepsilon^5))$$
$$= \frac{1}{2} + 2\varepsilon - 6\varepsilon^3 + 14\varepsilon^5 + o(\varepsilon^5) \quad \text{as} \quad \varepsilon \to 0+.$$

By the finite-dimensional Spectral Theorem we have $|A + \mathrm{i}B| = \sqrt{\lambda_1} R^{(t,1)} + \sqrt{\lambda_2} R^{(1-t,-1)}$, where

$$\sqrt{\lambda_1} = 1 + \varepsilon^2 + o(\varepsilon^3), \quad \sqrt{\lambda_2} = \varepsilon^2 + o(\varepsilon^3) \quad \text{as} \quad \varepsilon \to 0+$$

by Lemma 1 with $b = 1/2$ and via the relation $\varepsilon^k o(\varepsilon^m) = o(\varepsilon^{k+m})$ as $\varepsilon \to 0+$ for all $k, m \in \mathbb{N}$. Hence

$$\varphi(|A + \mathrm{i}B|) = \mathrm{tr}(S_\varphi |A + iB|) = \sqrt{\lambda_1}\mathrm{tr}(S_\varphi R^{(t,1)}) + \sqrt{\lambda_2}\mathrm{tr}(S_\varphi R^{(1-t,-1)})$$
$$= \sqrt{\lambda_1}\left(\left(\frac{1}{2} - s\right)t + \left(\frac{1}{2} + s\right)(1-t)\right) + \sqrt{\lambda_2}\left(\left(\frac{1}{2} - s\right)(1-t) + \left(\frac{1}{2} + s\right)t\right)$$
$$= \frac{1}{2} + \varepsilon^2 - 4s\varepsilon + 12s\varepsilon^3 + o(\varepsilon^3) \quad \text{as} \quad \varepsilon \to 0+.$$

Now the inequality $\varphi(A) \le \varphi(|A + \mathrm{i}B|)$ turns into

$$\frac{1}{2} - 2s\varepsilon \le \frac{1}{2} + \varepsilon^2 - 4s\varepsilon + 12s\varepsilon^3 + o(\varepsilon^3) \quad \text{as} \quad \varepsilon \to 0+.$$

It holds for all ε, $0 < \varepsilon < \frac{1}{2}$, for $s = 0$ only. The assertion is proved. □

Corollary 1. *For a positive normal linear functional φ on a von Neumann algebra $\mathcal{A}$ the following conditions are equivalent:*

(i) *φ is tracial;*

(ii) *$\varphi(|X - \mathrm{Re}X|) \le \varphi(|X - Y|)$ for all $X \in \mathcal{A}$ and $Y \in \mathcal{A}^{sa}$, where $\mathrm{Re}X = (X + X^*)/2$.*

Proof. (i) ⇒ (ii). By [30], for all $A, B \in \mathcal{A}$ there exist partial isometries $U, V \in \mathcal{A}$ such that

$$|A + B| \le U|A|U^* + V|B|V^*.$$

Hence $\varphi(|A \pm B|) \le \varphi(|A|) + \varphi(|B|)$ for all $A, B \in \mathcal{A}$. If $A = W|A|$ is the polar decomposition of the operator A then $W^*W \le I$, $|A^*| = W|A|W^*$ and $\varphi(|A^*|) = \varphi(W|A|W^*) = \varphi(|A|^{1/2}W^*W|A|^{1/2}) \le \varphi(|A|)$ by monotonicity of φ on the cone $\mathcal{A}^+$. By changing the roles of A and A^*, and taking into account the equality $(A^*)^* = A$, we obtain $\varphi(|A|) \le \varphi(|A^*|)$. Thus $\varphi(|A|) = \varphi(|A^*|)$ for every $A \in \mathcal{A}$. Under this fact and the triangle inequality, we have

$$\varphi\left(\left|X - \frac{1}{2}(X + X^*)\right|\right) = \frac{1}{2}\varphi(|X - X^*|) = \frac{1}{2}\varphi(|X - Y + Y - X^*|)$$

$$\le \frac{1}{2}(\varphi(|X-Y|)+\varphi(|(X-Y)^*|))=\varphi(|X-Y|).$$

(ii) $\Rightarrow$ (i). Let us put $X=\mathrm{i}A$ for $A\in\mathcal{A}^+$. Then $|X|=A$, $\mathrm{Re}X=0$ and for all $Y\in\mathcal{A}^{\mathrm{sa}}$ we have

$$\varphi(|X|)\le\varphi(|\mathrm{i}A-Y|)=\varphi(|A+\mathrm{i}Y|),$$

since $|\mathrm{i}A-Y|=|-\mathrm{i}(\mathrm{i}A-Y)|=|A+\mathrm{i}Y|$. Thus $\varphi(A)\le\varphi(|A+\mathrm{i}Y|)$ for all $A\in\mathcal{A}^+$, $Y\in\mathcal{A}^{\mathrm{sa}}$ and we apply Theorem 1. This completes the proof. □

Corollary 2 (cf. [8]). *Let φ be a positive normal linear functional on a von Neumann algebra $\mathcal{A}$. If $|\varphi(X)|\le\varphi(|X|)$ for all $X\in\mathcal{A}$ with $ReX\ge 0$ then φ is tracial.*

Proof. If $X\in\mathcal{A}$ and in the Cartesian decomposition $X=A+\mathrm{i}B$ the operator A is positive then $\varphi(A)\in\mathbb{R}^+$, $\varphi(B)\in\mathbb{R}$ and $\varphi(A)\le|\varphi(A+\mathrm{i}B)|$. Since by assumption $|\varphi(A+\mathrm{i}B)|\le\varphi(|A+\mathrm{i}B|)$, we have $\varphi(A)\le\varphi(|A+\mathrm{i}B|)$ for all $A\in\mathcal{A}^+$, $B\in\mathcal{A}^{\mathrm{sa}}$ and apply Theorem 1. □

Note that Corollary 2 strengthens the Gardner's characterization [8].

Corollary 3. *For a von Neumann algebra $\mathcal{A}$ the following conditions are equivalent:*

(i) *the algebra $\mathcal{A}$ is Abelian;*

(ii) $\varphi(A)\le\varphi(|A+\mathrm{i}B|)$ *for all normal states φ on $\mathcal{A}$ and $A\in\mathcal{A}^+$, $B\in\mathcal{A}^{\mathrm{sa}}$.*

(iii) $\varphi(|X-\mathrm{Re}X|)\le\varphi(|X-Y|)$ *for all normal states φ on $\mathcal{A}$* and $X\in\mathcal{A}$, $Y\in\mathcal{A}^{\mathrm{sa}}$.

Proof. (ii) $\Rightarrow$ (i). By Theorem 1 every normal state on $\mathcal{A}$ is tracial, i.e., $\varphi(XY)=\varphi(YX)$ for all $X,Y\in\mathcal{A}$. Since the set of all normal states separates the points of the algebra $\mathcal{A}$ [25, Chap. III, Theorem 2.4.5], the last condition implies that $XY=YX$ $(X,Y\in\mathcal{A})$, and thus the von Neumann algebra $\mathcal{A}$ is commutative.

For (iii) $\Rightarrow$ (i) we apply Corollary 1 instead of Theorem 1. □

Corollary 4. *Let φ be a normal semifinite weight on a von Neumann algebra $\mathcal{A}$ such that $\varphi(A)\le\varphi(|A+iB|)$ for all $A\in\mathcal{A}^+$ and $B\in\mathcal{A}^{sa}$ (or $\varphi(|X-ReX|)\le\varphi(|X-Y|)$ for all $X\in\mathcal{A}$ and $Y\in\mathcal{A}^{sa}$). Then φ is a trace.*

Proof. It follows by Theorem 1 (respectively, by Corollary 1) that for every projection $T\in\mathcal{A}^{\mathrm{pr}}$ with $\varphi(T)<\infty$ the reduced weight φ_T on the reduced algebra $T\mathcal{A}T$ is a trace. Hence φ is a trace by [31, Lemma 2]. □

For other trace characterizations see [29, 31–37] and references therein.

Theorem 2. *Let φ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{A}$. Then $\varphi(A)\le\varphi(|A+iB|)$ for all $A\in\mathcal{A}^+$* and $B\in\mathcal{A}^{\mathrm{sa}}$.

Proof. For every $X\in\mathcal{A}$, the generalised singular value function $\mu(X)$, given by $t\mapsto\mu(t,X)$ for $t\in(0,+\infty)$, is defined by the formula (see, e.g., [38, 39])

$$\mu(t,X)=\inf\{\|XP\|:\ P\in\mathcal{A}^{\mathrm{pr}},\ \ \varphi(P^\perp)\le t\}.$$

By Proposition 3 of [40] we have $\mu(t,A)\le\mu(t,|A+\mathrm{i}B|)$ for all $A\in\mathcal{A}^+$, $B\in\mathcal{A}^{\mathrm{sa}}$ and $t\in(0,+\infty)$. Therefore,

$$\varphi(A)=\int_0^{+\infty}\mu(t,A)dt\le\int_0^{+\infty}\mu(t,|A+\mathrm{i}B|)dt=\varphi(|A+\mathrm{i}B|).$$

The assertion is proved. □

Theorem 3. *Let $|||\cdot|||$ be a unitarily invariant norm on a unital C^*-algebra $\mathcal{A}$. Then $|||A|||\le|||A+iB|||$ for all $A\in\mathcal{A}^+$ and $B\in\mathcal{A}^{\mathrm{sa}}$.*

Proof. Recall that $|||Z|||=|||Z^*|||=||||Z||||$ for all $Z\in\mathcal{A}$. By [24, Theorem IX.7.1] we have

$$|||X-\mathrm{Re}X|||\le|||X-Y|||\quad\text{for all}\quad X\in\mathcal{A}\quad\text{and}\quad Y\in\mathcal{A}^{\mathrm{sa}}.$$

Let us put $X=\mathrm{i}A$ for $A\in\mathcal{A}^+$. Then $|X|=A$, $\mathrm{Re}X=0$ and for all $B\in\mathcal{A}^{\mathrm{sa}}$ we have

$$|||A|||=||||X||||=|||X|||=|||X-\mathrm{Re}X|||\le|||X-B|||=|||\mathrm{i}A-B|||=|||A+\mathrm{i}B|||,$$

since $|\mathrm{i}A-B|=|-\mathrm{i}(\mathrm{i}A-B)|=|A+\mathrm{i}B|$. Thus $|||A|||\le|||A+\mathrm{i}B|||$ for all $A\in\mathcal{A}^+$ and $B\in\mathcal{A}^{\mathrm{sa}}$. □

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