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Differences and Commutators of Idempotents in C^*-Algebras

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Abstract—We establish similarity between some tripotents and idempotents on a Hilbert space $\mathcal{H}$ and obtain new results on differences and commutators of idempotents P and Q. In the unital case, the difference $P-Q$ is associated with the difference $A_{P,Q}$ of another pair of idempotents. Let φ be a trace on a unital C^*-algebra $\mathcal{A}$, $\mathfrak{M}_\varphi$ be the ideal of definition of the trace φ. If $P-Q \in \mathfrak{M}_\varphi$, then $A_{P,Q} \in \mathfrak{M}_\varphi$ and $\varphi(A_{P,Q}) = \varphi(P-Q) \in \mathbb{R}$. In some cases, this allowed us to establish the equality $\varphi(P-Q)=0$. We obtain new identities for pairs of idempotents and for pairs of isoclinic projections. It is proved that each operator $A \in \mathcal{B}(\mathcal{H})$, $\dim \mathcal{H} = \infty$, can be presented as a sum of no more than 50 commutators of idempotents from $\mathcal{B}(\mathcal{H})$. It is shown that the commutator of an idempotent and an arbitrary element from an algebra $\mathcal{A}$ cannot be a nonzero idempotent. If $\mathcal{H}$ is separable and $\dim \mathcal{H} = \infty$, then each skew-Hermitian operator $T \in \mathcal{B}(\mathcal{H})$ can be represented as a sum $T = \sum_{k=1}^{4} [A_k, B_k]$, where $A_k, B_k \in \mathcal{B}(\mathcal{H})$ are skew-Hermitian.

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INTRODUCTION

Let P, Q be idempotents on a Hilbert space $\mathcal{H}$. Various properties (invertibility, Fredholm property, trace class property, positivity etc.) of the difference $X = P - Q$ have been studied in [1]–[6]. Any tripotent ($A = A^3$) is a difference $P-Q$ of some idempotents P and Q with $PQ = QP = 0$ [7, Proposition 1]. Hence tripotents inherit some of the properties of idempotents [8]. If X is a trace class operator, the traces of all odd degrees of X coincide:

$$\mathrm{tr}(P-Q) = \mathrm{tr}((P-Q)^{2n+1}) = \dim\ker(X-I) - \dim\ker(X+I) \in \mathbb{Z}, \tag{1}$$

here I is the identity operator on $\mathcal{H}$. If X is a compact operator, the right-hand side of (1) gives a natural "regularization" for the trace, showing that it always is an integer [9], [6]. In [10, Theorem 3], a C^*-analogue of the following statement is established: Let φ be a trace on a unital C^*-algebra $\mathcal{A}$, $\mathfrak{M}_\varphi$ be the ideal of definition of the trace φ, and $P, Q \in \mathcal{A}$ be tripotent; if $P - Q \in \mathfrak{M}_\varphi$, then $\varphi(P-Q) \in \mathbb{R}$.

Pairs of idempotents play important role in the Quantum Hall Effect [11]. For idempotents P, Q, R with trace class differences $P-Q$ and $Q-R$, the equality $\mathrm{tr}(P-Q) = \mathrm{tr}(P-R) + \mathrm{tr}(R-Q)$ together with (1) imply

$$\mathrm{tr}((P-Q)^3) = \mathrm{tr}((P-R)^3) + \mathrm{tr}((R-Q)^3). \tag{2}$$

Physical sense of additivity in (2) comes from interpretation of $\mathrm{tr}((P-Q)^3)$ as *the Hall conductance.* Additivity of (cubic) equation in (2) can be seen as a variant of the Ohm's law on additivity of conductance [12]. In [13, Theorem 1], a C^*-analogue of the Quantum Hall Effect is obtained and

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it is proved there that the trace of differences of a wide class of symmetries from a C^*-algebra is real [13, Corollaries 2 and 3]. For C^*-subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, we set

$$\mathcal{A}_0 = \left\{ X \in \mathcal{A} : \, X = \sum\nolimits_{n \geq 1} [X_n, \, X_n^*] \quad \text{for } (X_n)_{n \geq 1} \subset \mathcal{A} \right\},$$

where the series $\| \cdot \|$-converges. In [14, Theorem 2.6], it is proved that $\mathcal{A}_0$ coincides with the nullspace of all finite traces on $\mathcal{A}^{\mathrm{sa}}$; for a wide class of C^*-algebras, containing all W^*-algebras, it is sufficient to consider finite sums of the form [15]. If $P, Q \in \mathcal{A}^{\mathrm{id}}$, 1) $QP \in \mathcal{A}^{\mathrm{id}}$ if and only if $[P, \, Q]$ maps subspace $P\mathcal{H}$ into subspace $\operatorname{Ker} Q$ [16, Ch. II, Problem 241]; 2) P and Q are *equivalent* if and only if $P - Q = [X, Y]$ and $P + Q = XY + YX$ for some $X, Y \in \mathcal{A}$ [17, p. 97]. In [18], unital C^*-algebras without finite non-trivial traces are described in terms of finite sums of commutators.

In this article, we establish similarity between some tripotents and idempotents (Theorems 1 and 2). New results on differences and commutators of idempotents P and Q are obtained. In the unital case, the difference $P - Q$ is associated with the difference $A_{P,Q}$ of another pair of idempotents. If $P - Q \in \mathfrak{M}_\varphi$, then $A_{P,Q} \in \mathfrak{M}_\varphi$ and $\varphi(A_{P,Q}) = \varphi(P - Q) \in \mathbb{R}$ (Theorem 3). In some cases, this allowed us to establish the equality $\varphi(P - Q) = 0$ (Corollary 3). We obtain new identities for pairs of idempotents and for pairs of isoclinic projections (Lemma 6, Theorem 5). It is proved that each operator $A \in \mathcal{B}(\mathcal{H})$, $\dim \mathcal{H} = \infty$, can be presented as a sum of no more than 50 commutators of idempotents from $\mathcal{B}(\mathcal{H})$ (Theorem 6). If $\mathcal{A}$ is an algebra, $\{[P, X] : \; P \in \mathcal{A}^{\mathrm{id}}, \; X \in \mathcal{A}\} \cap \mathcal{A}^{\mathrm{id}} = \{0\}$ (Theorem 7). If $\mathcal{H}$ is separable and $\dim \mathcal{H} = \infty$, then each skew-Hermitian operator $T \in \mathcal{B}(\mathcal{H})$ can be represented as a sum $T = \sum_{k=1}^4 [A_k, B_k]$, where $A_k, B_k \in \mathcal{B}(\mathcal{H})$ are skew-Hermitian (Theorem 8). Let $n \in \mathbb{N}$ and $A, P \in \mathbb{M}_n(\mathbb{C})$ with $P = P^2$, $X = [A, P]$. Then (i) if $k \in \mathbb{N}$ is odd, X^k is a commutator; (ii) if $n \in \mathbb{N}$ is odd, $\det(X) = 0$ (Corollary 6).

1. DEFINITIONS AND NOTATION

For an algebra $\mathcal{A}$, by $\mathcal{A}^{\mathrm{id}}$ and $\mathcal{A}^{\mathrm{tri}}$ we will denote its subsets of idempotents ($P^2 = P$) and tripotents ($P^3 = P$) respectively. For $A, B \in \mathcal{A}$, define their commutator $[A, B] = AB - BA$. If $\mathcal{A}$ is unital, by I we denote the unit of algebra $\mathcal{A}$ and let $P^\perp = I - P$ for $P \in \mathcal{A}^{\mathrm{id}}$. The formula $S_P = 2P - I$ establishes a bijection between sets $\mathcal{A}^{\mathrm{id}}$ and $\mathcal{A}^{\mathrm{sym}}$.

A C^*-algebra is a complex Banach $*$-algebra $\mathcal{A}$ such that $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{A}$. For a C^*-algebra $\mathcal{A}$, by $\mathcal{A}^{\mathrm{pr}}$, $\mathcal{A}^{\mathrm{sa}}$ and $\mathcal{A}^+$ we will denote its subsets of projections ($P^2 = P = P^*$), Hermitian and positive elements respectively. Projections $P, Q \in \mathcal{A}$ are called *isoclinic* (with angle $\theta \in (0, \pi/2)$), if $PQP = \cos^2 \theta \, P$ and $QPQ = \cos^2 \theta \, Q$. If $A \in \mathcal{A}$, $|A| = \sqrt{A^*A} \in \mathcal{A}^+$. For a unital C^*-algebra $\mathcal{A}$, by $\mathcal{A}^{\mathrm{u}}$ and $\mathcal{A}^{\mathrm{inv}}$ we will denote its subsets of unitary and invertible elements respectively.

A W^*-algebra is a C^*-algebra $\mathcal{A}$ which has predual Banach space $\mathcal{A}_*$: $\mathcal{A} \simeq (\mathcal{A}_*)^*$. Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$, $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all linear bounded operators on $\mathcal{H}$. If $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$, then the projection $P \wedge Q$ is defined by the equality $(P \wedge Q)\mathcal{H} = P\mathcal{H} \cap Q\mathcal{H}$, and $P \vee Q = (P^\perp \wedge Q^\perp)^\perp$ projects on $\overline{\operatorname{lin}(P\mathcal{H} \cup Q\mathcal{H})}$. Any C^*-algebra can be represented as a C^*-subalgebra in $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (Gelfand–Naimark; see [19, Theorem 3.4.1]).

A *trace* on a C^*-algebra $\mathcal{A}$ is such a map $\varphi : \mathcal{A}^+ \to [0, +\infty]$ that $\varphi(X + Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda \varphi(X)$ for all $X, Y \in \mathcal{A}^+$, $\lambda \geq 0$ (wherein $0 \cdot (+\infty) \equiv 0$); $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{A}$. For a trace φ, define

$$\mathfrak{M}_\varphi^+ = \{X \in \mathcal{A}^+ : \; \varphi(X) < +\infty\}, \quad \mathfrak{M}_\varphi^{\mathrm{sa}} = \operatorname{lin}_{\mathbb{R}} \mathfrak{M}_\varphi^+, \quad \mathfrak{M}_\varphi = \operatorname{lin}_{\mathbb{C}} \mathfrak{M}_\varphi^+.$$

The restriction $\varphi|_{\mathfrak{M}_\varphi^+}$ can be correctly extended by linearity to a functional on $\mathfrak{M}_\varphi$ which we will denote by the same letter φ. A W^*-algebra is called *properly infinite*, if there is no nonzero normal finite trace on it.

2. DIFFERENCES AND COMMUTATORS OF IDEMPOTENTS ON C^*-ALGEBRAS

Let $\mathcal{A}$ be a W^*-algebra, $P, Q \in \mathcal{A}^{\rm pr}$ and $A = PQ$. Then there exists a symmetry $S \in \mathcal{A}^{\rm sa}$ such that $SAS^{-1} = A^*$ [20, Ch. 4, Exercise 4.4]. Let $A \in \mathcal{B}(\mathcal{H})$ be such that $SAS^{-1} = A^*$, where operator S is strongly invertible in the sense that zero does not lie in the closure of numerical image of S. Then A is similar to some $B \in \mathcal{B}(\mathcal{H})^{\rm sa}$ [21].

Lemma 1. *Let $\mathcal{A}$ be a unital C^*-algebra and $A \in \mathcal{A}$, $B \in \mathcal{A}^{\rm sa}$. If A and B are similar, A and A^* are also similar.*

Proof. Let $T \in \mathcal{A}^{\rm inv}$ be such that $A = T^{-1}BT$. Then $B = TAT^{-1}$ and for $S = T^*T \in \mathcal{A}^+$ we have

$$A^* = (T^{-1}BT)^* = T^*B(T^{-1})^* = T^*B(T^*)^{-1} = T^*TAT^{-1}(T^*)^{-1} = SAS^{-1}.$$

□

Theorem 1. *Let $A \in \mathcal{B}(\mathcal{H})^{\rm tri}$. Then A and A^* are similar.*

Proof. Due to [8, Theorem 3], any $A \in \mathcal{B}(\mathcal{H})^{\rm tri}$ is similar to some tripotent $B \in \mathcal{B}(\mathcal{H})^{\rm sa}$. Now, the desired statement follows from Lemma 1. □

The following lemma belongs to mathematical folklore.

Lemma 2. *Let $\mathcal{A}$ be a unital algebra and $P, Q \in \mathcal{A}^{\rm id}$. If $PQ = Q$ and $QP = P$ (respectively $PQ = P$ and $QP = Q$), P and Q are similar.*

Proof. Let

$$T = I - P + Q, \quad S = I + P - Q.$$

Then $TS = ST = I$ and $S = T^{-1}$. Obviously, $SPS^{-1} = Q$ (respectively $TPT^{-1} = Q$). □

In the settings of Lemma 2, we have $S_Q(P - Q)S_Q = Q - P$, and if $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$ with odd $n \in \mathbb{N}$, then the determinant $\det(P - Q) = 0$ due to the theorem on determinant of a product of matrices and due to the relation $\det(S_Q) \in \{-1, 1\}$.

Let $\mathcal{A}$ be a unital C^*-algebra and $P \in \mathcal{A}^{\rm id}$. There exists a unique decomposition $P = \widetilde{P} + Z$, where $\widetilde{P} \in \mathcal{A}^{\rm pr}$ and nilpotent $Z \in \mathcal{A}$ with $Z^2 = 0$, moreover, $Z\widetilde{P} = 0$, $\widetilde{P}Z = Z$ [22, Theorem 1.3].

Theorem 2 (cf. [23], Lemma 16)**.** *Let $\mathcal{A}$ be a unital C^*-algebra and $P \in \mathcal{A}^{\rm id}$, $P = \widetilde{P} + Z$ is the decomposition described above. Then P, $\widetilde{P}$, P^* are similar.*

Proof. Since $Z\widetilde{P} = 0$ and $\widetilde{P}Z = Z$, we have $P\widetilde{P} = \widetilde{P}$ and $\widetilde{P}P = P$. Hence, P and $\widetilde{P}$ are similar due to Lemma 2. As $\widetilde{P} \in \mathcal{A}^{\rm sa}$, idempotents P and P^* are similar due to Lemma 1. □

Corollary 1. Let $\mathcal{A}$ be a unital C^*-algebra. For $S \in \mathcal{A}$, the following conditions are equivalent:

(i) $S \in \mathcal{A}^{\rm sym}$;

(ii) $S = TUT^{-1}$ for some $T \in \mathcal{A}^{\rm inv}$ and $U \in \mathcal{A}^{\rm sa} \cap \mathcal{A}^{\rm u}$.

Proof. (i)⇒(ii) If $P \in \mathcal{A}^{\rm id}$, $P = T\widetilde{P}T^{-1}$ for some $T \in \mathcal{A}^{\rm inv}$ due to Theorem 2 or [23, Lemma 16]. Hence,

$$S_P = 2P - I = 2T\widetilde{P}T^{-1} - I = T(2\widetilde{P} - I)T^{-1},$$

i. e., we can take $U = 2\widetilde{P} - I$. □

Definition. Let $\mathcal{A}$ be a unital algebra and $P, Q \in \mathcal{A}^{\rm id}$. Let

$$A_{P,Q} = S_QPS_Q - S_PQS_P.$$

We have $A_{Q,P} = A_{P^\perp,Q^\perp} = -A_{P,Q}$, $A_{P^\perp,Q} = -A_{P,Q^\perp} = I - S_PQS_P - S_QPS_Q$ and $A_{P,Q}(P - Q) = (P - Q)A_{P,Q}$. Let $\mathcal{A}$ be a unital C^*-algebra and $P \in \mathcal{A}^{\mathrm{id}}$, $P = \widetilde{P} + Z$ be the decomposition described above. Then $A_{\widetilde{P},P} = 3P - 3\widetilde{P} = 3Z$.

Lemma 3. *Let J be an ideal in a unital algebra $\mathcal{A}$, $P, Q \in \mathcal{A}^{\mathrm{id}}$ and $\lambda, \mu \in \mathbb{C}$, $\lambda\mu \neq 0$, $\lambda \neq -\mu$. Then*

(i) *if $P - Q \in J$, $A_{P,Q} \in J$;*

(ii) *we have $P, Q \in J \Leftrightarrow \lambda P + \mu Q \in J$.*

Proof. (i) We have

$$A_{P,Q} = S_P(P - Q)S_P + S_Q(P - Q)S_Q - (P - Q) = 4QPQ - 4PQP + (P - Q). \tag{3}$$

In particular, $QPQ - PQP \in J$.

(ii), "$\Leftarrow$". We have

$$P = \frac{\mu}{\lambda(\lambda + \mu)} P(\lambda P + \mu Q)\Big(\frac{\lambda + \mu}{\mu} I - Q\Big) \in J.$$

□

It is seen from (3) that if $\{PQ, QP\} \cap \{0\} \neq \varnothing$ (or $\{P, Q\} \cap \{I\} \neq \varnothing$), $A_{P,Q} = P - Q$.

Theorem 3. *Let φ be a trace on a unital C^*-algebra $\mathcal{A}$. If $P, Q \in \mathcal{A}^{\mathrm{id}}$ and $P - Q \in \mathfrak{M}_\varphi$, then $A_{P,Q} \in \mathfrak{M}_\varphi$ and $\varphi(A_{P,Q}) = \varphi(P - Q) \in \mathbb{R}$.*

Proof. Recall that $\mathfrak{M}_\varphi$ is an ideal in $\mathcal{A}$, moreover, $\varphi(XY) = \varphi(YX)$ for all $X \in \mathfrak{M}_\varphi$, $Y \in \mathcal{A}$ [19, Ch. 6, Exercise 6]. Due to item (i) of Lemma 3, we obtain $A_{P,Q} \in \mathfrak{M}_\varphi$. Since

$$\varphi(S_P(P - Q)S_P) = \varphi(S_Q(P - Q)S_Q) = \varphi(P - Q),$$

we have $\varphi(A_{P,Q}) = \varphi(P - Q) \in \mathbb{R}$ due to linearity of the extension of φ to $\mathfrak{M}_\varphi$, (3) and due to [10, Theorem 3]. □

Corollary 2. In the settings of item (i) of Theorem 3, for any $n \in \mathbb{N}$ we have

$$\varphi(A_{P,Q}^{2n+1}) = \varphi(A_{P,Q}) = \varphi(P - Q) \in \mathbb{R}.$$

Proof. For any $n \in \mathbb{N}$, we obtain from [13, Theorem 1] and (1) that

$$\varphi(A_{P,Q}^{2n+1}) = \varphi(A_{P,Q}) = \varphi(4QPQ - 4PQP + P - Q) = \varphi(P - Q) \in \mathbb{R},$$

since $QPQ - PQP \in \mathfrak{M}_\varphi$ and $\varphi(QPQ - PQP) = 0$ (see step 2 of the proof of [13, Theorem 1]). □

Note that item (i) of the following theorem generalizes item (i) of [24, Theorem 3.2].

Theorem 4. *Let φ be a trace on a C^*-algebra $\mathcal{A}$.*

(i) *If $X \in \mathcal{A}^{\mathrm{tri}}$, $Y \in \mathcal{A}$ and $[X, Y] \in \mathfrak{M}_\varphi$, then $\varphi([X, Y]) = 0$.*

(ii) *If $X, Y \in \mathcal{A}$ and $[X, Y] \in \mathfrak{M}_\varphi$, then $[X^k, Y^n] \in \mathfrak{M}_\varphi$ for all $k, n \in \mathbb{N}$.*

(iii) *If $X, Y \in \mathcal{A}$ and $X - Y \in \mathfrak{M}_\varphi$, then $[X^k, Y^n] \in \mathfrak{M}_\varphi$ and $\varphi([X^k, Y^n]) = 0$ for all $k, n \in \mathbb{N}$.*

Proof. (i) Step 1. Let $X \in \mathcal{A}^{\text{id}}$. Since

$$XY - 2XYX + YX = X[X, Y] - [X, Y]X \in \mathfrak{M}_\varphi,$$

the statement follows from the representation

$$[X, Y] = X(XY - 2XYX + YX) - (XY - 2XYX + YX)X$$

and linearity of the extension of φ to $\mathfrak{M}_\varphi$.

Step 2. Let $X \in \mathcal{A}^{\text{tri}}$ and $X = P - Q$ with $P, Q \in \mathcal{A}^{\text{id}}$ and $PQ = QP = 0$ [7, Proposition 1]. Then $X^2 = P + Q \in \mathcal{A}^{\text{id}}$ and

$$[P, Y] + [Q, Y] = [X^2, Y] = X[X, Y] + [X, Y]X \in \mathfrak{M}_\varphi.$$

By the condition, $[P, Y] - [Q, Y] = [X, Y] \in \mathfrak{M}_\varphi$. From the two last relations, we have $[P, Y], [Q, Y] \in \mathfrak{M}_\varphi$, and due to step 1 and linearity of the extension of φ to $\mathfrak{M}_\varphi$, we obtain

$$\varphi([X, Y]) = \varphi([P, Y]) - \varphi([Q, Y]) = 0 - 0 = 0.$$

(ii) Let us use the method of mathematical induction. For all $k \geq 2$, we have

$$[X^k, Y] = X[X^{k-1}, Y] + [X, Y]X^{k-1} \in \mathfrak{M}_\varphi.$$

For all $n \geq 2$, we obtain

$$[X^k, Y^n] = Y[X^k, Y^{n-1}] + [X^k, Y]Y^{n-1} \in \mathfrak{M}_\varphi.$$

(iii) Step 1. With the help of mathematical induction, we will show that $X^k - Y^k \in \mathfrak{M}_\varphi$ for all $k \in \mathbb{N}$. Suppose that $X^{k-1} - Y^{k-1} \in \mathfrak{M}_\varphi$. Then

$$X^k - Y^k = X^{k-1}(X - Y) + (X^{k-1} - Y^{k-1})Y \in \mathfrak{M}_\varphi,$$

which was required.

Step 2. From the representation

$$X^kY^n - Y^nX^k = (X^k - Y^k)Y^n - Y^n(X^k - Y^k),$$

it follows that $[X^k, Y^n] \in \mathfrak{M}_\varphi$ and

$$\varphi([X^k, Y^n]) = \varphi((X^k - Y^k)Y^n) - \varphi(Y^n(X^k - Y^k)) = 0$$

for all $k, n \in \mathbb{N}$ due to linearity of the extension of φ to $\mathfrak{M}_\varphi$. □

In particular, if $X \in \mathcal{A}$, $P \in \mathcal{A}^{\text{id}}$ and $XP - PXP \in \mathfrak{M}_\varphi$, then $\varphi(XP - PXP) = 0$ due to the equality $XP - PXP = [XP, P]$ (see item (i) of Theorem 4).

Example 1. Let $\mathcal{A}$ be an algebra and $P, Q \in \mathcal{A}^{\text{id}}$, $PQ = Q$ and $QP = P$. Then $PQP = P$ and $QPQ = Q$; we have $(P + Q)^k = 2^k(P + Q)$ for all $k \in \mathbb{N}$ and $(P - Q)^2 = 0$. Hence, due to the theorem on the determinant of a product of matrices, for $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$ we obtain $\det(P + Q) = \det(P - Q) = 0$.

For idempotents

$$P = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{M}_3(\mathbb{C})^{\text{id}},$$

we have $PQP = P$ and $QPQ = Q$, however $\{PQ, QP\} \cap \{P, Q\} = \varnothing$.

Lemma 4. *Let $\mathcal{A}$ be a unital algebra and $P, Q \in \mathcal{A}^{\mathrm{id}}$, $\lambda \in \mathbb{C} \setminus \{0\}$. Let*

$$A = (1-\lambda)P + (\lambda^{-1} - \lambda - 1 + \lambda^2)PQ + \lambda QP + (\lambda^2 - \lambda^4)Q, \quad B = (1-\lambda)Q + (2\lambda^{-1} - 1)PQ.$$

If $PQP = \lambda^2 P$ and $QPQ = \lambda^2 Q$, then idempotents P and A (respectively Q and B) are similar. We have $(\lambda P - \lambda^{-1}QP)^2 = (\lambda Q - \lambda^{-1}PQ)^2 = 0$.

Proof. Let

$$T = I + \lambda^{-1}PQ - \lambda Q, \quad S = I - \lambda^{-1}PQ + \lambda Q.$$

Then $TS = ST = I$ and $S = T^{-1}$. We have $SPS^{-1} = A$ and $TQT^{-1} = B$, hence, $A, B \in \mathcal{A}^{\mathrm{id}}$. The equalities $(\lambda P - \lambda^{-1}QP)^2 = (\lambda Q - \lambda^{-1}PQ)^2 = 0$ can be easily checked. □

Corollary 3. Let $\mathcal{A}$ be a unital algebra and $P, Q \in \mathcal{A}^{\mathrm{id}}$. If $PQP = P$ and $QPQ = Q$, then idempotents P and QP (respectively Q and PQ) are similar. We have $(P - QP)^2 = (Q - PQ)^2 = 0$.

In the settings of Lemma 4, we have $A_{P,Q} = (1 - 4\lambda^2)(P - Q)$, and if φ is a trace on a unital C^*-algebra $\mathcal{A}$ and $P - Q \in \mathfrak{M}_\varphi$, then $\varphi(P - Q) = 0$. If $\mathcal{A}$ is a unital $*$-algabra and $P, Q \in \mathcal{A}^{\mathrm{id}}$, then $PQ = Q \Leftrightarrow Q^{*\perp}P^{*\perp} = P^{*\perp}$.

Lemma 5. *If $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ and $PQP = P$, then $QP = P$, i. e., $P \leq Q$.*

Proof. Since $Q \cdot PQP = QP \cdot QP = QP$, we have

$$(P - QP)^2 = Q^\perp PQ^\perp P = 0.$$

Multiply both sides of this relation on the left by projection P, to get $(PQ^\perp P)^2 = 0$. Since $PQ^\perp P \in \mathcal{B}(\mathcal{H})^+$, we have $0 = PQ^\perp P = |Q^\perp P|^2$, i. e., $|Q^\perp P| = 0$ and $Q^\perp P = 0$. □

Example 2. Let $\mathcal{A}$ be a unital C^*-algebra, projections $P, Q \in \mathcal{A}^{\mathrm{pr}}$ be isoclinic with some angle $\theta \in (0, \pi/2)$. Then $(\cos^2\theta\, P - QP)^2 = 0$ and

$$A_{P,Q} = (1 - 4\cos^2\theta)(P - Q), \tag{4}$$

for $\theta = \pi/3$ we have $A_{P,Q} = 0$. Recall [25, Ch. 2, § 10, item 10.5, (iii)] that

$$P \vee Q = \frac{1}{\sin^2\theta}(P - Q)^2. \tag{5}$$

Hence, $P \vee Q \in \mathcal{A}$,

$$A^2_{P,Q} = (1 - 4\cos^2\theta)^2 \sin^2\theta\, P \vee Q,$$

$$\sin(P - Q) = \frac{\sin(\sin\theta)}{\sin\theta}(P - Q), \quad \cos(P - Q) = I + (\cos(\sin\theta) - 1)\, P \vee Q,$$

$$\sinh(P - Q) = \frac{\sinh(\sin\theta)}{\sin\theta}(P - Q), \quad \cosh(P - Q) = I + (\cosh(\sin\theta) - 1)\, P \vee Q$$

and $\exp(P - Q) = \sinh(P - Q) + \cosh(P - Q)$. The relation

$$(P - Q)^4 = (P - Q)^2 - |PQ - QP|^2 \tag{6}$$

(see the proof of [10, Proposition 1]) and (5) give

$$|[P, Q]| = \sin\theta\cos\theta\, P \vee Q.$$

If J is a left (or right) ideal in $\mathcal{A}$ and $P - Q \in J$, then $P \vee Q \in J$ due to equality (5). Hence, projections $P = P \vee Q \cdot P$ and $Q = P \vee Q \cdot Q$ lie in J. It is clear that

$$P - Q \in J \Leftrightarrow (P - Q)^2 \in J \Leftrightarrow |[P, Q]| \in J \Leftrightarrow P \vee Q \in J \Leftrightarrow P, Q \in J.$$

If $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$, we obtain from the theorem on determinant of a product of matrices and from (5) that

$$\det(P - Q) = \begin{cases} 0, & \text{if } P \vee Q \neq I; \\ \pm \sin^n \theta, & \text{if } P \vee Q = I. \end{cases}$$

Corollary 4. Let φ be a trace on a unital C^*-algebra $\mathcal{A}$ and projections $P, Q \in \mathcal{A}^{\mathrm{pr}}$ be isoclinic with some angle $\theta \in (0, \pi/2)$. If $P - Q \in \mathfrak{M}_\varphi$, then $P, Q \in \mathfrak{M}_\varphi$, and from Theorem 3 and equality (4) we obtain $0 = \varphi(P - Q) = \varphi(P) - \varphi(Q)$. From equality (5) we have $\varphi(P \vee Q) = \varphi(P) + \varphi(Q) = 2\varphi(P)$.

Lemma 6. *Let $\mathcal{A}$ be an algebra and $P, Q \in \mathcal{A}^{\mathrm{id}}$. Then*

(i) $(P - Q)^4 + (P + Q)^4 = 2(P + Q)^2 + 2(PQ + QP)^2$;

(ii) $(P - Q)^2 + (P + Q)^2 = 2(P + Q)$;

(iii) *if $\mathcal{A}$ is unital,* $[P, Q] = (I - P - Q)(P - Q) = -(P - Q)(I - P - Q)$.

Theorem 5. *Let $\mathcal{A}$ be a C^*-algebra and projections $P, Q \in \mathcal{A}^{\mathrm{pr}}$ be isoclinic with some angle $\theta \in (0, \pi/2)$. Then $\sin^4 \theta \, P \vee Q + (P + Q)^4 = (2 + \cos^2 \theta)(P + Q)^2$, where $(P + Q)^2 = 2(P + Q) - \sin^2 \theta \, P \vee Q$.*

The proof follows from Lemma 6 and equality (5).

Lemma 7. (i) *If $\mathcal{A}$ is a properly infinite W^*-algebra, then each commutator $[A, B]$ $(A, B \in \mathcal{A})$ can be represented as a sum of no more than* 25 *commutators of idempotents from $\mathcal{A}$.*

(ii) *If $\mathcal{H}$ is separable and* $\dim \mathcal{H} = \infty$, *then each commutator $[A, B]$ of operators $A, B \in \mathcal{B}(\mathcal{H})^{\mathrm{sa}}$ with $\|A\| < 1$, $\|B\| < 1$, can be represented as a sum of no more than* 2025 *commutators of projections from $\mathcal{B}(\mathcal{H})$.*

Proof. (i) Due to [26, Theorem 4], we have

$$A = P_1 + \ldots + P_5, \quad B = Q_1 + \ldots + Q_5$$

with some $P_k, Q_k \in \mathcal{A}^{\mathrm{id}}$, $k = 1, \ldots, 5$.

(ii) If $\mathcal{H}$ is separable and $\dim \mathcal{H} = \infty$, each operator $T \in \mathcal{B}(\mathcal{H})^{\mathrm{sa}}$ with $\|T\| < 1$ can be represented as

$$T = 5(P_1 + P_2 + P_3 + P_4) - 5P_5 - 8P_6 - 12P_7$$

with $P_1, \ldots, P_7 \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ [27, Remark 4]. □

Theorem 6. *Each operator $A \in \mathcal{B}(\mathcal{H})$, $\dim \mathcal{H} = \infty$, can be represented as a sum of no more than* 50 *commutators of idempotents from $\mathcal{B}(\mathcal{H})$.*

Proof. Any operator in an infinite-dimensional Hilbert space $\mathcal{H}$ can be represented as a sum of two commutators [28, Corollary 2 from Problem 186]. Now we apply item (i) of Lemma 7, since $\mathcal{B}(\mathcal{H})$ is a properly infinite W^*-algebra. □

Theorem 7. *If $\mathcal{A}$ is an algebra, $\{[P, X] : P \in \mathcal{A}^{\mathrm{id}}, X \in \mathcal{A}\} \cap \mathcal{A}^{\mathrm{id}} = \{0\}$. Generally speaking, $\{[P, Q] : P, Q \in \mathcal{A}^{\mathrm{id}}\} \cap \mathcal{A}^{\mathrm{tri}} \neq \{0\}$.*

Proof. Let $P \in \mathcal{A}^{\mathrm{id}}$, $X \in \mathcal{A}$ and

$$[P, X]^2 = [P, X]. \tag{7}$$

Multiply both sides of (7) on the left and on the right by idempotent P, to get

$$PXPXP = PX^2P. \tag{8}$$

Then, multiply both sides of (7) on the right by P, and take into account (8), we obtain $PXP = XP$. Multiply both sides of (7) on the left by P, and take into account (8), we obtain $PX = PXP$. Hence, $[P, X] = 0$ and $\{[P, X] : \ P \in \mathcal{A}^{\mathrm{id}}, \ X \in \mathcal{A}\} \cap \mathcal{A}^{\mathrm{id}} = \{0\}$.

Numbers

$$a = \frac{\sqrt{5} - 1}{2}, \quad b = \sqrt{a - a^2} = \sqrt{\sqrt{5} - 2}$$

satisfy the condition $2a - b^2 = 1$. In algebra $\mathcal{A} = \mathbb{M}_2(\mathbb{C})$, for idempotents

$$P = \begin{pmatrix} 1 & b^{-1} \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} a & b \\ b & 1 - a \end{pmatrix}$$

we have $[P, Q]^2 = \mathrm{diag}(1, 1) = I$, i.e., $[P, Q] \in \mathcal{A}^{\mathrm{sym}} \subset \mathcal{A}^{\mathrm{tri}} \setminus \{0\}$. □

Any operator from $\mathcal{B}(\mathcal{H})$, $\dim \mathcal{H} = \infty$, can be represented as a finite sum of pair-wise products of projections ([29]; [30], a theorem). Hence, any skew-Hermitian operator ($A^* = -A$) from $\mathcal{B}(\mathcal{H})$ can be represented as a finite sum of commutators of projections [24, Theorem 5.1]. The following theorem was announced by the first author without proof in [24, p. 12, Statement I].

Theorem 8. *If $\mathcal{H}$ is separable and $\dim \mathcal{H} = \infty$, any skew-Hermitian operator $T \in \mathcal{B}(\mathcal{H})$ can be represented as a sum $T = \sum_{k=1}^{4} [A_k, B_k]$, where $A_k, B_k \in \mathcal{B}(\mathcal{H})$ are skew-Hermitian.*

Proof. We will use [28, Corollary 2 from Problem 186]: any operator $T \in \mathcal{B}(\mathcal{H})$ can be represented as a sum of two commutators: $T = [A, B] + [C, D]$ with $A, B, C, D \in \mathcal{B}(\mathcal{H})$. Let $T = -T^*$ and $T = [A, B] + [C, D]$. Then

$$T = \frac{T - T^*}{2} = \frac{AB - BA + A^*B^* - B^*A^* + CD - DC + C^*D^* - D^*C^*}{2}. \tag{9}$$

For any $Y \in \mathcal{B}(\mathcal{H})$, operators $Y - Y^*$, $i(Y + Y^*)$ are skew-Hermitian, where $i \in \mathbb{C}$ and $i^2 = -1$. It is easy to prove that

$$[A - A^*, B - B^*] + [i(B + B^*), i(A + A^*)] = 2AB - 2BA + 2A^*B^* - 2B^*A^*.$$

Thus,

$$\frac{AB - BA + A^*B^* - B^*A^*}{2} = \left[\frac{A - A^*}{2}, \frac{B - B^*}{2}\right] + \left[\frac{i(B + B^*)}{2}, \frac{i(A + A^*)}{2}\right], \tag{10}$$

$$\frac{CD - DC + C^*D^* - D^*C^*}{2} = \left[\frac{C - C^*}{2}, \frac{D - D^*}{2}\right] + \left[\frac{i(D + D^*)}{2}, \frac{i(C + C^*)}{2}\right]. \tag{11}$$

Substitute the right-hand sides of (10) and (11) into (9) to complete the proof. □

Corollary 5. If $\mathcal{H}$ is separable and $\dim \mathcal{H} = \infty$, any skew-Hermitian operator $T \in \mathcal{B}(\mathcal{H})$ can be represented as a sum $T = \sum_{k=1}^{4} [C_k, D_k]$, *where* $C_k, D_k \in \mathcal{B}(\mathcal{H})^{\mathrm{sa}}$.

Proof. Let $C_k = iB_k$, $D_k = iA_k$ for $k = 1, 2, 3, 4$. □

If $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$, then (6) implies (see also [4, Proposition 3])

$$|PQ - QP|^2 = (P - Q)^2 - (P - Q)^4 \leq (P - Q)^2. \tag{12}$$

Theorem 9. *Let φ be a faithful trace on a W^*-algebra $\mathcal{A}$, $A \in \mathcal{A}$ and $P \in \mathcal{A}^{\mathrm{id}}$. For $X = [A, P]$, we have $S_P X = -X S_P$. If $X^k \in \mathfrak{M}_\varphi$ for some odd $k \in \mathbb{N}$, $\varphi(X^k) = 0$. If, moreover, $P = P^*$, then $[|X|, P] = 0$, and for $A \in \mathcal{A}^{\mathrm{pr}}$ with $X^2 \in \mathfrak{M}_\varphi$ we have $\varphi(X^2) = 0 \Leftrightarrow X = 0$.*

Proof. It is clear that $XS_P = -S_PX$. For $U \in \mathcal{A}$ and $V \in \mathfrak{M}_\varphi$, we have $\varphi(UV) = \varphi(VU)$ (see [19, Ch. 6, Exercise 6]). Thus, if $X^k \in \mathfrak{M}_\varphi$ for some odd $k \in \mathbb{N}$, then $\varphi(X^k) = 0$ (cf. [5, Theorem 2.26]). If $P = P^*$, then $X^*S_P = -S_PX^*$ and $S_PX^*S_P = -X^*$. Hence, $|X|^2 = S_P|X|^2S_P$, i. e., $|X|^2S_P = S_P|X|^2$ and $|X|^2P = P|X|^2$. Now, due to the spectral theorem, we have $|X|P = P|X|$.

Let $A, P \in \mathcal{A}^{\mathrm{pr}}$, $X = [A, P]$ and $X^2 \in \mathfrak{M}_\varphi$ with $\varphi(X^2) = 0$. Since $X^2 = -|X|^2$, from (12) we get

$$0 = \varphi(X^2) = \varphi(-|X|^2) = -\varphi(|X|^2) = -\varphi((A-P)^2 - (A-P)^4). \tag{13}$$

Since $(A-P)^2 - (A-P)^4 \geq 0$ (recall that $\|A-P\| \leq 1$) and since trace φ is faithful, from (13) we have $(A-P)^2 - (A-P)^4 = 0$, i. e., $(A-P)^2 = |A-P|^2 \in \mathcal{A}^{\mathrm{pr}}$. Hence, operator $U = A-P$ is a partial isometry on $\mathcal{H}$. Hence, $UU^*U = U$ [28, Corollary 3 from Problem 98]. From the equality $(A-P)^3 = A-P$, we get $PAP = APA$. Hence, $PAP \leq A$ and $AP = PA$ due to [31, Proposition 2.1]. □

Corollary 6. Let $n \in \mathbb{N}$ and $A, P \in \mathbb{M}_n(\mathbb{C})$ with $P = P^2$, $X = [A, P]$.

(i) If $k \in \mathbb{N}$ is odd, X^k is a commutator.

(ii) If $n \in \mathbb{N}$ is odd, $\det(X) = 0$.

Proof. It is known that for $T \in \mathbb{M}_n(\mathbb{C})$, the following conditions are equivalent: 1) T is unitarily equivalent to a matrix with zero diagonal; 2) trace $\mathrm{tr}(T) = 0$; 3) T is a commutator; 4) $\mathrm{tr}(|I + zT|) \geq n$ for all $z \in \mathbb{C}$. The proof of equivalency 1)⇔2) see in [16, Ch. II, Problem 209], equivalency 2)⇔3) is proved in [28, Problem 182], equivalency 2)⇔4) is established in [32, Theorem 4.8].

(i) Use equivalency 2)⇔3).

(ii) Since $S_P^2 = I$ and $\det(S_P) \in \{-1, 1\}$ due to the theorem on determinant of a product of matrices, we apply this theorem to the equality $S_PX = -XS_P$ with $X = [A, P]$. □

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