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On the Representation of Semigroup C^*-Algebra as a Crossed Product

E. V. Lipacheva[a,b,*]

[a] *Lobachevsky Institute of Mathematics and Mechanics, Kazan (Volga Region) Federal University, Kazan, 420008 Russia*

[b] *Chair of Higher Mathematics, Kazan State Power Engineering University, Kazan, 420066 Russia*

**e-mail: elipacheva@gmail.com*

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Abstract—The semidirect product $\mathbb{Z} \rtimes_{\varphi} \mathbb{Z}^{\times}$ of the additive group $\mathbb{Z}$ of integers and the multiplicative semigroup $\mathbb{Z}^{\times}$ of nonzero integers with respect to a homomorphism φ from $\mathbb{Z}^{\times}$ to the semigroup of endomorphisms of the group $\mathbb{Z}$ is constructed. It is shown that the semigroup $\mathbb{Z} \rtimes_{\varphi} \mathbb{Z}^{\times}$ is a normal extension of the Cartesian product $\mathbb{Z} \times \mathbb{N}$ by means of the residue class group modulo two, where $\mathbb{N}$ is the multiplicative semigroup of all natural numbers. Reduced semigroup C^*-algebras for the semigroup $\mathbb{Z} \times \mathbb{N}$ are studied. A dynamical system for the semigroup C^*-algebra of the semigroup $\mathbb{Z} \times \mathbb{N}$ is considered, and its covariant representation is defined. The semigroup C^*-algebra for the semigroup $\mathbb{Z} \rtimes_{\varphi} \mathbb{Z}^{\times}$ is represented as a crossed product of the C^*-algebra for the $\mathbb{Z} \times \mathbb{N}$ semigroup with the residue class group modulo two.

Keywords: dynamical system, covariant representation, normal extension of semigroups, semidirect product of semigroups, reduced semigroup C^*-algebra, crossed product of a C^*-algebra with a group

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The article considers the representation of a reduced semigroup C^*-algebra as a crossed product of its subalgebra with some group. To do this, we introduce a C^*-dynamical system for the semigroup C^*-algebra and study the corresponding crossed product.

The reduced semigroup C^*-algebras are operator algebras generated by the left regular representations of cancellative semigroups. Such algebras were first studied in by L. Coburn [1, 2]. He considered a reduced semigroup C^*-algebra for the additive semigroup of nonnegative integers. Then R. Douglas [3] studied semigroup C^*-algebras for subsemigroups in the additive group of all real numbers. The results of articles [1–3] were generalized by H. Murphy [4, 5]. He considered C^*-algebras for positive cones in arbitrary ordered groups. The study of semigroup C^*-algebras was continued by A. Nica [6], M. Laca and I. Raeburn [7], and other authors (see, e.g., [8] and references therein).

J. Cuntz [9] studied the C^*-algebra corresponding to the so-called $ax + b$-semigroup $\mathbb{N} \rtimes \mathbb{N}^{\times}$ over the set of natural numbers $\mathbb{N}$. This work initiated the study of semigroup C^*-algebras corresponding to $ax + b$-semigroups over rings of integers in number fields, and C^*-algebras over rings (see, e.g., [10, 11]). In particular, parallels were drawn there between C^*-algebras over rings and reduced semigroup C^*-algebras. In [12], further results in this direction were obtained.

The present paper is a continuation of the studies begun in [13–16] to reveal the relationship between normal extensions of semigroups and the corresponding reduced semigroup C^*-algebras. In particular, in [15], examples of normal extensions of semigroups by means of commutative groups were given. In [16], trivial extensions of semigroups and the corresponding semigroup C^*-algebras were studied. In [17], a reduced semigroup C^*-algebra was considered for the semidirect product $\mathbb{Z} \rtimes_{\varphi} \mathbb{Z}^{\times}$, the construction of which is described in Section 1 of this article. We represent this algebra as a crossed product of its C^*-subalgebra and the residue class group modulo two.

Crossed products are widely used in the theory of operator algebras and in mathematical physics (see, e.g., the monograph [18] and the references therein). It should be noted that the C^*-crossed product generalizes the notion of a group C^*-algebra $C^*(G)$ for a locally compact group G. Namely, this algebra is a crossed product $\mathbb{C} \rtimes G$ of the field $\mathbb{C}$ of complex numbers and a group G ([18], p. 54). The general construction of the crossed product $\mathcal{A} \rtimes_\alpha G$ as a C^*-enveloping algebra of the Banach *-algebra $L^1(G, \mathcal{A})$ of the classes of function on a group G with values in the C^*-algebra $\mathcal{A}$ is given, e.g., in ([19], § 7.6).

In Section 1 of the article, the necessary information from the theory of semigroup extensions is given and the semigroup $\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times$ and the corresponding extension of the semigroup $\mathbb{Z} \times \mathbb{N}$ by means of the group $\mathbb{Z}_2$ are constructed. In Section 2, reduced semigroup C^*-algebras $C_r^*(\mathbb{Z} \times \mathbb{N})$ and $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ are studied. For this, we consider a dynamical system for the semigroup C^*-algebra $C_r^*(\mathbb{Z} \times \mathbb{N})$ and its covariant representation. A proposition is formulated that the C^*-algebra $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ is isomorphic to the crossed product of the C^*-algebra $C_r^*(\mathbb{Z} \times \mathbb{N})$ with the residue group $\mathbb{Z}_2$.

1. EXTENSION OF SEMIGROUPS BY MEANS OF THE GROUP $\mathbb{Z}_2$

First, we give definitions of normal and Schreier extensions of semigroups by means of arbitrary groups.

Let S and L be semigroups and G be a group with an identity e. Let there be an injective semigroup homomorphism $\tau : S \to L$ and a surjective semigroup homomorphism $\sigma : L \to G$. A triple (L, τ, σ) is called the *normal extension* of a semigroup S by means of a group G if $\tau(S)$ is the complete preimage of the identity of the group G, i.e., if

$$\sigma^{-1}(e) = \tau(S).$$

The semigroup L itself is also called the extension of a semigroup S by means of a group G. General definitions of semigroup extensions can be found in [20, 21].

A normal extension (L, τ, σ) of a semigroup S is called a *Schreier extension* if the following condition is satisfied: there is a subset $X \subset L$ such that $X \cap \sigma^{-1}(g) = \{x_g\}$ for any $g \in G$ and each element $y \in L$ is uniquely represented as $y = \tau(a)x_g$ for some $a \in S$ and $g \in G$.

Let $\mathbb{Z}$ be an additive group of integers and $\mathbb{Z}^\times := \mathbb{Z} \setminus \{0\}$ be a multiplicative semigroup of integers without zero. We define a representation $\varphi : \mathbb{Z}^\times \to \mathrm{End}(\mathbb{Z})$ of the semigroup $\mathbb{Z}^\times$ into the semigroup of endomorphism of $\mathbb{Z}$ as follows:

$$\varphi(m) := \begin{cases} \mathrm{id}, & \text{if} \quad m > 0; \\ \mathrm{inv}, & \text{if} \quad m < 0, \end{cases}$$

where $m \in \mathbb{Z}^\times$, id is the identity endomorphism, and inv is the inverse endomorphism; i.e., $\mathrm{inv}(n) = -n$, $n \in \mathbb{Z}$. Consider the semidirect product $\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times$. This is a semigroup under multiplication

$$(m, n)(k, l) = (m + \varphi(n)(k), nl),$$

where $m, k \in \mathbb{Z}$, $n, l \in \mathbb{Z}^\times$. It is easy to see that $\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times$ is a cancellative semigroup with an identity (0, 1).

Let $\mathbb{Z} \times \mathbb{N}$ be the Cartesian product of the additive group of integers and the multiplicative semigroup of natural numbers. This is a semigroup with identity (0, 1) under multiplication

$$(m, n)(k, l) = (m + k, nl),$$

where $m, k \in \mathbb{Z}$, $n, l \in \mathbb{N}$.

Let us define a mapping

$$\tau : \mathbb{Z} \times \mathbb{N} \to \mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times : (m, n) \mapsto (m, n).$$

Obviously, τ is an injective semigroup homomorphism.

Let $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} = \{0,1\}$ be the residue class group modulo two. We define another mapping by the following formula:

$$\sigma : \mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times \to \mathbb{Z}_2 : (m,n) \mapsto \begin{cases} 0, & \text{if} \quad n > 0; \\ 1, & \text{if} \quad n < 0. \end{cases}$$

It is easy to see that σ is a surjective homomorphism of semigroups. Thus, we have a short exact sequence

$$(0,1) \to \mathbb{Z} \times \mathbb{N} \xrightarrow{\ \tau\ } \mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times \xrightarrow{\ \sigma\ } \mathbb{Z}_2 \to 0.$$

Moreover, we can formulate the following proposition.

Proposition 1. The triple $(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times, \tau, \sigma)$ is a Schreier extension of the semigroup $\mathbb{Z} \times \mathbb{N}$ by the group $\mathbb{Z}_2$.

2. REDUCED SEMIGROUP C^*-ALGEBRA $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$

Recall the definition of a reduced semigroup C^*-algebra.

Let S be a discrete semigroup with left cancellation. We introduce the Hilbert space $l^2(S)$. Recall that it consists of square summable complex-valued functions on S. We denote e_a, $a \in S$, a function of the space $l^2(S)$ defined by the formulas $e_a(b) = 1$ if $a = b$, and $e_a(b) = 0$ if $a \neq b$, for any $b \in S$. Then, the set of functions $\{e_a \mid a \in S\}$ is an orthonormal basis of the Hilbert space $l^2(S)$.

In the algebra of all bounded linear operators $B(l^2(S))$ on the space $l^2(S)$, consider the C^*-subalgebra $C_r^*(S)$ generated by the set of isometries $\{T_a \mid a \in S\}$, where $T_a(e_b) = e_{ab}$, $a, b \in S$. It is called a *reduced semigroup C^*-algebra*.

Consider the reduced semigroup C^*-algebras $C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times)$ and $C_r^*(\mathbb{Z} \times \mathbb{N})$. The sets of generating isometric operators of these C^*-algebras are denoted $\{\tilde{T}_{(m,n)} \mid m \in \mathbb{Z}, n \in \mathbb{Z}^\times\} \subset B(l^2(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times))$ and $\{T_{(m,n)} \mid m \in \mathbb{Z}, n \in \mathbb{N}\} \subset B(l^2(\mathbb{Z} \times \mathbb{N}))$, respectively.

By Proposition 1 and Theorem 3.1 [15], we have the following proposition.

Proposition 2. There is a unique isometric *-homomorphism of semigroup C^*-algebras:

$$\phi : C_r^*(\mathbb{Z} \times \mathbb{N}) \to C_r^*(\mathbb{Z} \rtimes_\varphi \mathbb{Z}^\times) : T_{(m,n)} \mapsto \tilde{T}_{\tau(m,n)} = \tilde{T}_{(m,n)},$$

where $m \in \mathbb{Z}, n \in \mathbb{N}$.

To formulate the main results of this article, we need the definitions of a C^*-dynamical system, a covariant representation, and a crossed product constructed from a given dynamical system.

Let $\mathcal{A}$ be a C^*-algebra, G be a locally compact group, and $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be a continuous group homomorphism. Then, the triple $(\mathcal{A}, G, \alpha)$ is called a C^*-dynamical system or simply a dynamical system. A *covariant representation* of the dynamical system $(\mathcal{A}, G, \alpha)$ is such a pair (π, u), consisting of a representation $\pi : \mathcal{A} \to B(H)$ and a unitary representation $u : G \to B(H)$ for the same Hilbert space H that

$$\pi(\alpha_g(a)) = u(g)\pi(a)u(g)^*$$

for any $a \in \mathcal{A}$, $g \in G$ ([18], pp. 43–44).

In [22], the definition of the *crossed product* $\mathcal{A} \rtimes_\alpha G$ corresponding to a dynamical system $(\mathcal{A}, G, \alpha)$ is given in terms of the universal approach. It is shown that there is a one-to-one correspondence between the representations of the C^*-algebra $\mathcal{A} \rtimes_\alpha G$ and the covariant representations of the dynamical system $(\mathcal{A}, G, \alpha)$.

Let us further consider the semigroup C^*-algebra $C_r^*(\mathbb{Z} \times \mathbb{N})$. We introduce the notation: $U_m := T_{(m,1)}$, $m \in \mathbb{Z}$, and $V_n := T_{(0,n)}$, $n \in \mathbb{N}$. It is easy to see that $U_m^* = U_{-m}$; therefore, U_m is a unitary operator for any $m \in \mathbb{Z}$.

Lemma. *The semigroup C^*-algebra $C_r^*(\mathbb{Z} \times \mathbb{N})$ coincides with the C^*-subalgebra in $B(l^2(\mathbb{Z} \times \mathbb{N}))$, generated by the set of unitary operators $\{U_m \mid m \in \mathbb{Z}\}$ and the set of isometries $\{V_n \mid n \in \mathbb{N}\}$.*

This lemma allows one to define the C^*-dynamical system $(C_r^*(\mathbb{Z}\times\mathbb{N}),\mathbb{Z}_2,\alpha)$, where $\alpha:\mathbb{Z}_2\to \mathrm{Aut}(C_r^*(\mathbb{Z}\times\mathbb{N}))$ is a group homomorphism such that $\alpha_0=\mathrm{id}$ and α_1 is correctly defined by the action on the generating elements of the C^*-algebra $C_r^*(\mathbb{Z}\times\mathbb{N})$ by the formulas

$$\alpha_1(U_m)=U_m^*,\quad \alpha_1(V_n)=V_n$$

for any $m\in\mathbb{Z}$, $n\in\mathbb{N}$.

Proposition 2 implies that there exists a representation

$$\phi: C_r^*(\mathbb{Z}\times\mathbb{N})\to B(l^2(\mathbb{Z}\rtimes_\varphi\mathbb{Z}^\times)): T_{(m,n)}\mapsto \tilde{T}_{(m,n)}$$

of semigroup C^*-algebra on the Hilbert space $l^2(\mathbb{Z}\rtimes_\varphi\mathbb{Z}^\times)$. We define a unitary representation $u:\mathbb{Z}_2\to B(l^2(\mathbb{Z}\rtimes_\varphi\mathbb{Z}^\times))$ of the group $\mathbb{Z}_2$ by setting

$$u(0)=I,\ u(1)=\tilde{T}_{(0,-1)}.$$

We have the following theorem.

Theorem 1. *Pair* (ϕ, u) *is a covariant representation of the dynamical system* $(C_r^*(\mathbb{Z}\times\mathbb{N}),\mathbb{Z}_2,\alpha)$.

The next theorem states that the reduced semigroup C^*-algebra $C_r^*(\mathbb{Z}\rtimes_\varphi\mathbb{Z}^\times)$ coincides up to isomorphism with the crossed product corresponding to the C^*-dynamical system $(C_r^*(\mathbb{Z}\times\mathbb{N}),\mathbb{Z}_2,\alpha)$.

Theorem 2. *There is an isomorphism of* C^**-algebras*

$$C_r^*(\mathbb{Z}\rtimes_\varphi\mathbb{Z}^\times)\cong C_r^*(\mathbb{Z}\times\mathbb{N})\rtimes_\alpha\mathbb{Z}_2.$$

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CONFLICT OF INTEREST

The author declares that she has no conflicts of interest.

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