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New insights into superintegrability from unitary matrix models

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ABSTRACT

Some eigenvalue matrix models possess an interesting property: one can manifestly define the basis where all averages can be explicitly calculated. For example, in the Gaussian Hermitian and rectangular complex models, averages of the Schur functions are again expressed through the Schur functions. However, so far this property remains restricted to very particular (e.g. Gaussian) measures. In this paper, we extend this observation to unitary matrix integrals, where one could expect that this restriction is easier to lift. We demonstrate that this is indeed the case, only this time the Schur averages are linear combinations of the Schur functions. Full factorization to a single item in the sum appears only on the Miwa locus, where at least one half of the time-variables is expressed through matrices of the same size. For unitary integrals, this is a manifestation of the de Wit-t'Hooft anomaly, which prevents the answer to be fully analytic in the matrix size N. Once achieved, this understanding can be extended back to the Hermitian model, where the phenomenon looks very similar: beyond Gaussian measures superintegrability requires an additional summation.

1. Introduction

Matrix models play a special role in theoretical physics, because they are the simplest prototypes of the full string theory. As any integrals, their partition functions are invariant under the change of integration variables, but, in this case, the corresponding Ward identities are especially simple, and acquire the form of linear Virasoro-like constraints, which further imply a hidden integrability, i.e. bilinear Hirota-type identities [1]. In fact, this is not the end of the story: the most interesting classes of matrix models possess an even stronger feature of *superintegrability* [16], that is, in a certain full basis all the correlation functions are explicitly calculable (see many examples in [2–23], and also some preliminary results in [25–29]). This resembles emergence of closed orbits for the motion in the Coulomb and harmonic oscillator fields, and the name *superintegrability* is borrowed from those well-known examples. So far, this property was found for peculiar classes of matrix models, see [24] for a brief review. The unitary matrix models did not belong to the list in that summary, and in this paper we explain what is the difference. In unitary models, additional time-variables $\bar{p}_k$, which parameterize the background potential, appear on equal footing with the ordinary couplings p_k, which are used to describe arbitrary correlators. This gives a chance to go beyond Gaussian models, towards arbitrary Dijkgraaf-Vafa phases and even further.

In this paper, we study this possibility and demonstrate that, in general, the averages are not just factorized, but are represented as series, which get reduced to single terms only at the Miwa locus. Moreover, the formulas are not fully analytic in the matrix size N, a well known phenomenon for unitary integrals, sometime called the de Wit-t'Hooft anomaly. It also appears straightforward to describe not only the character phase, to which our formalism is *a priori* tuned, but also the "opposite" Kontsevich phase, i.e. a kind of a non-perturbative expansion.

After this peculiarity of superintegrability is understood, we can return to the Hermitian matrix model and obtain the sum, which generalizes the usual factorized expressions of Gaussian averages. A way to cleverly handle such sums and to restore analyticity in N remains for future analysis.

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Notation. In this letter, one of the main objects are the Schur functions $S_R(x_i)$ [33], which are symmetric functions of variables x_i, and are labelled by partitions R. One can equivalently consider them as graded polynomials of power sums $p_k := \sum_i x_i^k$, we use the notation $S_R\{p_k\}$ in this case. For an $N\times N$ matrix X with the eigenvalues x_i so that $p_k = \mathrm{Tr}\, X^k$ we will also use the notation $S_R[X]$. For the partition R, we denote through l_R the number of its parts. Definitions of other functions used in the letter can be found in the short Appendix.

2. The unitary matrix model

The main quantity of our interest in this paper is the partition function of unitary matrix model [30,31]

$$Z_N\{p,\bar p\} := \int [DU] \exp\left(\sum_k \frac{p_k \mathrm{Tr}\, U^k}{k} + \frac{\bar p_k \mathrm{Tr}\left(U^\dagger\right)^k}{k}\right) \tag{1}$$

where U is $N\times N$ unitary matrix and $[dU]$ denotes the Haar measure on unitary matrices. Since the integrand depends only on invariant quantities, one can integrate over angular variables to get an eigenvalue model reducing the partition function to an integral over the eigenvalues z_i of U, which are unimodular numbers, $\bar z_i = \frac{1}{z_i}$. We normalize the measure DU in a way that it includes the normalization factor (the volume of the unitary group $U(N)$) emerging from this integration over the angular variables. Then, the eigenvalue representation of the partition function gets the form [32]

$$Z_N\{p,\bar p\} = \prod_{i=1}^N \oint_{|z_i|=1} \frac{dz_i}{z_i} \left|\Delta(z)\right|^2 \exp\left(\sum_k \frac{p_k z_i^k}{k} + \frac{\bar p_k}{k z_i^k}\right) = \det_{1\le i,j\le N} C_{i-j}\{p,\bar p\} \tag{2}$$

where $\Delta(z) := \prod_{i<j}(z_i - z_j)$ is the Vandermonde determinant, and the moment matrix

$$C_k\{p,\bar p\} := \oint \frac{dz}{z} z^k \exp\left(\sum_k \frac{p_k z^k}{k} + \frac{\bar p_k}{k z^k}\right) \tag{3}$$

Making use of the formula

$$\exp\left(\sum_k \frac{p_k z^k}{k}\right) = \sum_m S_{[m]}\{p\} z^m \tag{4}$$

we can shift the $p, \bar p$-dependence to symmetric Schur polynomials $S_{[m]}$:

$$\begin{aligned} Z_N\{p,\bar p\} &= \det_{1\le i,j\le N} C_{i-j}\{p,\bar p\} = \det_{1\le i,j\le N} \sum_{k,l} S_{[k]}\{p\} S_{[l]}\{\bar p\} C_{i-j+k-l}\{0,0\} = \\ &= \det_{1\le i,j\le N} \sum_{k,l} S_{[j+k]}\{p\} S_{[i+l]}\{\bar p\} C_{k-l}\{0,0\} \end{aligned} \tag{5}$$

3. Superintegrability of the unitary matrix model

In fact, since $C_k\{0,0\} = \delta_{k,0}$, one can further rewrite this sum as (see [34, Eq.(7)])

$$\det_{1\le i,j\le N} \sum_k S_{[j+k]}\{p\} S_{[i+k]}\{\bar p\} \overset{CB}{=} \sum_{k_1>k_2>\ldots>k_N} \det_{1\le i,j\le N} S_{[k_i+j]}\{p\} \cdot \det_{1\le i,j\le N} S_{[k_i+j]}\{\bar p\} = \sum_{R:\ l_R\le N} S_R\{p\} S_R\{\bar p\} \tag{6}$$

where we used the Cauchy-Binet formula (CB),

$$\det_{1\le i,j\le N}\left(\sum_k A_{ik} B_{kj}\right) \overset{CB}{=} \sum_{k_1>k_2>\ldots>k_N} \det_{1\le i,j\le N} A_{ik_j} \cdot \det_{1\le i,j\le N} B_{ik_j} \tag{7}$$

In the last transition in (9) we changed the variables $k_i := R_i - i$, what allows us to represent the sum as going over partitions R. We also used the first Jacobi-Trudi identity

$$S_R\{p\} = \det_{i,j} S_{[R_i - i + j]}\{p\} \tag{8}$$

to substitute determinants of symmetric Schur polynomials as generic Schur functions. Thus, we finally obtain

$$Z_N\{p,\bar p\} = \int [DU] \exp\left(\sum_k \frac{p_k \mathrm{Tr}\, U^k}{k} + \frac{\bar p_k \mathrm{Tr}\left(U^\dagger\right)^k}{k}\right) = \sum_{R:\ l_R\le N} S_R\{p\} S_R\{\bar p\} \tag{9}$$

Eq. (9) implies that

$$\left\langle \exp\left(\sum_k P_k \mathrm{Tr}\, U^k\right)\right\rangle = \sum_{R:\ l_R\le N} S_R\{p+P\} S_R\{\bar p\} = \sum_Q S_Q\{P\} \sum_{R:\ l_R\le N} S_{R/Q}\{p\} S_R\{\bar p\} \tag{10}$$

and since

$$\left\langle \exp\left(\sum_k P_k \mathrm{Tr}\, U^k\right)\right\rangle = \sum_Q S_Q\{P\}\left\langle S_Q[U]\right\rangle \tag{11}$$

this allows us to get an average of the Schur function:

$$\left\langle S_Q[U]\right\rangle = \sum_{R:\, l_R\le N} S_{R/Q}\{p\}S_R\{\bar p\} \tag{12}$$

Likewise,

$$\boxed{\left\langle S_{Q_1}[U]S_{Q_2}[U^\dagger]\right\rangle = \sum_{R:\, l_R\le N} S_{R/Q_1}\{p\}S_{R/Q_2}\{\bar p\}} \tag{13}$$

This sum goes over infinite set of arbitrary large partitions R, still it is always convergent. This is not that surprising if one recalls the Cauchy identity for the skew Schur functions

$$\sum_R S_{R/Q_1}\{p\}S_{R/Q_2}\{\bar p\} = \exp\left(\sum_k \frac{p_k\bar p_k}{k}\right)\sum_\sigma S_{Q_1/\sigma}\{\bar p\}S_{Q_2/\sigma}\{p\} \tag{14}$$

This identity expresses the infinite sum over R at the l.h.s. via the finite sum over σ at the r.h.s., and in (13) this infinite sum is further restricted to $l_R \le N$.

4. Reduction to special Miwa locus

Let us note that the restriction of sums over partitions, $l_R \le N$ like that in formula (9) can be automatically fulfilled of one performs a Miwa transform of the variables p_k and restricts the number of Miwa variables by N:

$$\bar p_k = \sum_{i=1}^N x_i^k \tag{15}$$

This is what happens in the Hermitian matrix model: a similar sum for its partition function [13] is not specifically restricted, since the summand contains $S_R\{\bar p_k = N\}$ instead of $S_R\{\bar p_k\}$ with arbitrary $\bar p_k$. The condition $\bar p_k = N$ means there are only N Miwa variables, $x_i = 1$, $i \le N$, and this condition emerges automatically in the model. On contrary, such a condition is not obligatory imposed in the unitary model case, which gives a more generic example, but at the price of explicit restriction on the summation domain. However, if one restricts $\bar p_k$ to only N (arbitrary) Miwa variables (15), it gives

$$Z_N\{p,x\} = \sum_R S_R\{p\}S_R(x_i) = \exp\left(\sum_{k,i}\frac{p_k x_i^k}{k}\right) \tag{16}$$

One can also realize this Miwa transform by an $N\times N$ matrix X such that $\bar p_k = \mathrm{Tr}\, X^k$:

$$Z_N\{p,X\} = \int [DU]\exp\left(\sum_k \frac{p_k \mathrm{Tr}\, U^k}{k} + \frac{\mathrm{Tr}\, X^k \mathrm{Tr}\left(U^\dagger\right)^k}{k}\right) = \sum_R S_R\{p\}S_R[X] = \exp\left(\sum_k \frac{p_k \mathrm{Tr}\, X^k}{k}\right) \tag{17}$$

and similarly for (13). In particular, one can generate from this formula arbitrary correlators of positive powers of U in this restricted background of U^{-1} (it can be also obtained from (12) using (14)):

$$\left\langle S_Q[U]\right\rangle = S_Q[X]\exp\left(\sum_k \frac{p_k \mathrm{Tr}\, X^k}{k}\right) \tag{18}$$

or

$$\boxed{\left\langle\!\left\langle S_Q[U]\right\rangle\!\right\rangle = S_Q[X]} \tag{19}$$

for the normalized correlators. This brings us back to the factorized form of the Schur averages without summation like the Hermitian and complex model cases [13].

In fact, one can naturally consider even a more restrictive case of p_k also parameterized by N Miwa variables: $p_k = \sum_i y_i^k$ (or $p_k = \mathrm{Tr}\, Y^k$), it gives

$$Z_N\{y,x\} = \sum_R S_R(y_i)S_R(x_i) = \prod_{i,j}^N \frac{1}{1-x_i y_j} \tag{20}$$

Thus, the restriction to N Miwa variables makes the expressions too trivial.

5. A more general model: switching on background fields

Consider now a more general model with the partition function depending on two external $N \times N$ matrices A and B:

$$Z_N\{A, B, p, \bar{p}\} := \int [DU] \exp\left(\sum_k \frac{p_k \mathrm{Tr}\,(AU)^k}{k} + \frac{\bar{p}_k \mathrm{Tr}\,\left(U^\dagger B\right)^k}{k} \right) \tag{21}$$

In fact, this model depends on the matrix of product AB only, this follows from the change of variables $U \to BU$ and invariance of the Haar measure under this transformation.

In order to solve this model, we use the Cauchy formula in order to expand the r.h.s. into the sum of $S_R\{p\}S_Q\{\bar{p}\}$ and integrate the coefficients of expansion over U using the formula [33]

$$\int [DU] S_R[AU] S_Q[U^\dagger B] = \frac{S_R[AB]}{S_R\{N\}} \delta_{R,Q} \tag{22}$$

In fact, this formula can be used as an alternative to the calculation in (9). Its application to (21) provides the following result:

$$Z_N\{A, B, p, \bar{p}\} = \sum_{P,Q} S_R\{p\} S_Q\{\bar{p}\} \frac{S_R[AB]}{S_R\{N\}} \delta_{R,Q} = \sum_{R:\, l_R \le N} S_R\{p\} S_R\{\bar{p}\} \frac{S_R[AB]}{S_R\{N\}} \tag{23}$$

which reduces to (9) at $A = B = I$. Similarly to our calculation in the previous subsection, one can immediately obtain now the corresponding correlation functions:

$$\boxed{\left\langle S_{Q_1}[U] S_{Q_2}[U^\dagger] \right\rangle = \sum_{R:\, l_R \le N} S_{R/Q_1}\{p\} S_{R/Q_2}\{\bar{p}\} \frac{S_R[AB]}{S_R\{N\}}} \tag{24}$$

This expression looks like having poles at integer values of N, since

$$S_R\{N\} = \prod_{i,j \in R} (N - i + j) \cdot S_R\{p_k = \delta_{k,1}\} \tag{25}$$

However, it turns out that these poles do not emerge due to the restriction of the sum to the partitions of lengths not larger than N. Indeed, as follows from (25), the pole in N that may come from the diagram R is at most at $l_R - 1$. Hence, the restriction means that contributes only the diagrams that do not give rise to a pole at a given N. This simultaneously means that there is no analytic expression in N possible for the correlators in this model. This is called de Wit – t'Hooft anomaly [35].

6. BGW model

Choosing $p_k = \delta_{k,1}$, $\bar{p}_k = \delta_{k,1}$ gives us the Brezín-Gross-Witten (BGW) model [36] in the character phase, i.e. perturbatively in AB [37]. The BGW partition function looks like [37,38]

$$Z_N^{BGW}\{A\} := \int [DU] e^{\mathrm{Tr}\, A(U+U^{-1})} = \sum_{R:\, l_R \le N} \frac{S_R\{\delta_{k,1}\}^2}{S_R\{N\}} S_R[A^2] \tag{26}$$

and the correlators are

$$\boxed{\left\langle S_{Q_1}[U] S_{Q_2}[U^\dagger] \right\rangle = \sum_{R:\, l_R \le N} S_{R/Q_1}\{\delta_{k,1}\} S_{R/Q_2}\{\delta_{k,1}\} \frac{S_R[A^2]}{S_R\{N\}}} \tag{27}$$

Using the formula that expresses the skew Schur functions via the shifted Schur functions S^*_Q at the locus $p_k = \delta_{k,1}$, [39],

$$S_{R/Q}\{\delta_{k,1}\} = S_R\{\delta_{k,1}\} \cdot S^*_Q(R_i) \tag{28}$$

one can also rewrite this average in the form

$$\left\langle S_{Q_1}[U] S_{Q_2}[U^\dagger] \right\rangle = \sum_{R:\, l_R \le N} S^*_{Q_1}(R_i) S^*_{Q_2}(R_i) \frac{S_R\{\delta_{k,1}\}^2}{S_R\{N\}} S_R[A^2] \tag{29}$$

The quantity $S_R\{\delta_{k,1}\}$ is often denoted by d_R.

7. BGW model in Kontsevich phase

An interesting question is how to get an answer in the Kontsevich case [37], i.e. the integral

$$Z_N^{BGW}\{A\} := \int [DU] e^{\mathrm{Tr}\, A(U+U^{-1})} \tag{30}$$

at large A. For the sake of simplicity, from now on, we assume A is a Hermitian matrix with positive trace. In order for this integral expansion at large A to start from 1, we use a peculiar normalization of the integral [37]

$$Z_N^{BGW}\{A\}_+ := e^{-2\mathrm{Tr}\,A}\sqrt{\frac{\det(A\otimes A^2 + A^2\otimes A)}{(\det A)^N}}\int [DU]e^{\mathrm{Tr}\,A(U+U^{-1})} \tag{31}$$

Hereafter, we label the partition function $Z_N^{BGW}\{A\}$ at large values of A, i.e. in the Kontsevich phase, by the subscript $+$, and, at small values, i.e. in the character phase and without the peculiar normalization factor, by the subscript $-$.

7.1. The case of $N = 1$

It is instructive to begin from the $N = 1$ example, where the BGW partition function reduces to a (modified) Bessel function with well known expansions in both positive (character phase) and negative (Kontsevich phase) powers of λ.

Indeed, in this case,

$$Z_{N=1}^{BGW}\{A\}_- = \int\limits_0^{2\pi}\frac{d\phi}{2\pi}e^{2A\cos\phi} = I_0(2A) \tag{32}$$

and the series in A for the modified Bessel function [40] gives

$$Z_{N=1}^{BGW}\{A\}_- = \sum_{k=0}\frac{A^{2k}}{k!^2} \tag{33}$$

which coincides with (26) at $N = 1$.

Similarly, using the asymptotic expansion of the modified Bessel function at large A [40], one obtains

$$Z_{N=1}^{BGW}\{A\}_+ = 2\sqrt{A\pi}e^{-2A}I_0(2A) = \sum_{k=0}\frac{(2k)!^2}{k!^3}\left(\frac{1}{64A}\right)^k = \sum_{k=0}\frac{(-1)^k}{k!}\frac{\Gamma(1/2+k)}{\Gamma(1/2-k)}\left(\frac{1}{4A}\right)^k \tag{34}$$

However, the main point is not yet seen at the level of $N = 1$.

7.2. The case of $N = 2$: Q Schur versus Schur functions

The most interesting aspect of the story is that expansions at the two phases are drastically different: one, in the character phase, is in Schur functions, while another one, in the Kontsevich phase, is reduced to a smaller set of Q-Schur functions [33,41]. To see this difference, we should proceed to $N = 2$. Then, in the character phase, the expansion of the BGW partition function is

$$Z_{N=2}^{BGW}\{A\}_- = 1 + \frac{1}{2}(a_1^2 + a_2^2) + \frac{1}{12}(a_1^4 + 4a_1^2a_2^2 + a_2^4) = 1 + \frac{1}{2}p_1 + \frac{1}{6}p_1^2 - \frac{1}{12}p_2 \tag{35}$$

depends on *all* $p_k = a_1^{2k} + a_2^{2k}$ and can be expanded into the basis of the Schur functions. At the same time, in the Kontsevich phase (see (37)-(39) below),

$$\begin{aligned} Z_{N=2}^{BGW}\{A\}_+ = \frac{\psi_1(a_1)\psi_2(a_2) - \psi_1(a_2)\psi_2(a_1)}{a_1 - a_2} &= 1 + \frac{1}{16}\left(\frac{1}{a_1} + \frac{1}{a_2}\right) + \frac{9}{512}\left(\frac{1}{a_1^2} + \frac{2}{a_1a_2} + \frac{1}{a_2^2}\right) + \ldots = \\ &= 1 + \frac{1}{16}\tilde{p}_1 + \frac{9}{512}\tilde{p}_1^2 + \ldots \end{aligned} \tag{36}$$

the dependence on $\tilde{p}_2 := a_1^{-2} + a_2^{-2}$ disappears. Looking at higher terms in the expansion, one observes that actually *all* the variables with even indices, $\tilde{p}_{2k}$ drop out. This implies that, in the Kontsevich phase, one has to use a restricted basis of symmetric polynomials that depend only on p_k with odd k. Such a basis is known, these are Q Schur functions [33,41].

7.3. Generic integer N

The simplest way to obtain the expansion of $Z_N^{BGW}\{A\}_+$ at generic N is to use a determinant representation [37]:

$$Z_N^{BGW}\{A\}_+ = \frac{\det_{i,j}\psi_i(a_j)}{\Delta(a)} \tag{37}$$

where

$$\psi_i(a) := 2\sqrt{\pi}a^{i-1/2}e^{-2a}I_{i-1}(2a) \tag{38}$$

and we denoted through a_i the eigenvalues of the matrix A. As we already pointed out, an important property of (37) is that it is a function of traces of only odd degrees of the matrix A: $\tilde{p}_{2k-1} = \mathrm{Tr}\,A^{-2k+1}$. Now, using the asymptotics of the modified Bessel function [40]

$$I_n(2a) = \frac{e^{2a}}{2\sqrt{\pi a}} \sum_{k=0} \frac{(-1)^k}{k!} \frac{\Gamma(n+1/2+k)}{\Gamma(n+1/2-k)} \left(\frac{1}{4A}\right)^k \tag{39}$$

one obtains that

$$\boxed{Z_N^{BGW}\{A\}_+ = \frac{\det_{i,j} \psi_i(a_j)}{\Delta(a)} = \sum_{R\in SP} \left(\frac{1}{32}\right)^{|R|} Q_R\{\mathrm{Tr}\, A^{-k}\} \frac{Q_R\{\delta_{k,1}\}^3}{Q_{2R}\{\delta_{k,1}\}^2}} \tag{40}$$

where Q_R are the Q Schur functions, and SP means strict partitions, i.e. those with all lines of distinct lengths. This formula was conjectured in [42] and later proved in [43].

8. Generalized Itzykson-Zuber model

One can also consider a generalized Itzykson-Zuber integral:

$$Z_N^{IZ}\{A, B, p\} := \int [DU] \exp\left(\sum_k \frac{p_k \mathrm{Tr}\,(AUBU^\dagger)^k}{k}\right) \tag{41}$$

This model can be also solved in the same way using the formula for the integral [33]

$$\int [DU] S_R[AUBU^\dagger] = \frac{S_R[A] S_R[B]}{S_R\{N\}} \tag{42}$$

which is a kind of a "dual" to (22). Applying the Cauchy identity (14) to (41), we obtain

$$Z_N^{IZ}\{A, B, p\} = \sum_R S_R\{p\} \int [DU] S_R[AUBU^\dagger] = \sum_{R:\, l_R \le N} S_R\{p\} \frac{S_R[A] S_R[B]}{S_R\{N\}} \tag{43}$$

Using the same trick as before, one can obtain the correlators in the form

$$\boxed{\left\langle S_Q[AUBU^\dagger]\right\rangle^{IZ} := \int [DU] S_Q[AUBU^\dagger] \exp\left(\sum_k \frac{p_k \mathrm{Tr}\,(AUBU^\dagger)^k}{k}\right) = \sum_{R:\, l_R \le N} S_{R/Q}\{p\} \frac{S_R[A] S_R[B]}{S_R\{N\}}} \tag{44}$$

In fact, this sum is assumed to be automatically reduced to $l_R \le N$, since N is also the size of the background matrices A and B, and two zeroes in the numerator overweight a single zero in denominator. Hence, the restriction on summation can be omitted.

9. Back to the case of Hermitian model

In the case of Hermitian model [31,32],

$$\mathcal{Z}_N\{p\} := \int \rho(H) dH \exp\left(\sum_k \frac{p_k \mathrm{Tr}\, H^k}{k}\right) = \prod_{i=1}^N \int \rho(h_i) dh_i \cdot \Delta^2(h) \exp\left(\sum_k \frac{p_k h_i^k}{k}\right) = \det_{1\le i,j\le N} \mathcal{C}_{i+j-2}\{p\} \tag{45}$$

where $\rho(h)$ is a measure, and

$$\mathcal{C}_k\{p\} := \int h^k \rho(h) dh \exp\left(\sum_k \frac{p_k h^k}{k}\right) \tag{46}$$

one could make just the same calculation as in (9), see sec. 2.2 of [34]:

$$\begin{aligned}
Z_N\{p\} = \det_{1\le i,j\le N} \mathcal{C}_{i+j-2}\{p, \bar{p}\} = (-1)^{\frac{N(N-1)}{2}} \det_{1\le i,j\le N} \mathcal{C}_{N-i+j-1}\{p_k\} = \\
= (-1)^{\frac{N(N-1)}{2}} \det_{1\le i,j\le N} \sum_k S_{[k]}\{p_k\} \mathcal{C}_{N-i+j+k-1}\{0\} = \\
= (-1)^{\frac{N(N-1)}{2}} \det_{1\le i,j\le N} \sum_k S_{[k+i]}\{p_k\} \mathcal{C}_{N+j+k-1}\{0\} \overset{CB}{=} \\
\overset{CB}{=} (-1)^{\frac{N(N-1)}{2}} \sum_{k_1>k_2>\ldots>k_N} \det_{1\le i,j\le N} S_{[k_i+j]}\{p_k\} \cdot \det_{1\le i,j\le N} \mathcal{C}_{N+j+k_i-1}\{0\} = \\
= (-1)^{\frac{N(N-1)}{2}} \sum_{R:\, l_R\le N} \underbrace{\det_{1\le i,j\le N} \mathcal{C}_{N-i+j+R_i-1}\{0\}}_{\mathcal{C}_R} \cdot S_R\{p_k\}
\end{aligned} \tag{47}$$

The problem is that, in this case, there are no negative degrees of the matrix in the potential in (45) (otherwise, the matrix integral is not well-defined), and $\mathcal{C}_R$ is now a sophisticated function of R:

$$\mathcal{C}_R = \int x^{N-i+j+R_i-1}\rho(h)dh \tag{48}$$

Moreover, the factor $\rho(h)$ can not be ignored: one can not put $\rho(h) = 1$, as in (9), because then the integral over the eigenvalue h diverges. In fact, $\mathcal{C}_R$ can be explicitly calculated for the Gaussian measure $\rho(h) = e^{-h^2/2}$ and is expressed through the Schur functions [13]. Moreover, as we already mentioned in sec. 4, this Gaussian $\mathcal{C}_R$ explicitly vanishes for $l_R > N$, hence the sum in (47) can be extended to all R, so that particular averages get factorized. This is how the superintegrability phenomenon looks in the Hermitian case. The situation is similar in the complex rectangular model with Gaussian measure [13].

In the unitary case, one does not need $\rho(h) \neq 1$, but the answer involves a non-trivial restriction $l_R \le N$ in the sum beyond the restricted Miwa locus of sec. 4. Still, this extension is quite explicit and easy to handle, unlike (47) with non-Gaussian ρ.

Note that we have come closer to the situation with the Hermitian model by restricting $\bar p_k$ to a peculiar Miwa locus (15) in sec. 4 above: then unrestricted p_k can be used to generate arbitrary correlators of positive powers of U in a restricted background of U^{-1}, (19). Not surprisingly, further restriction of p_k to a similar peculiar Miwa locus makes the partition function elementary and is not interesting. An advantage of the unitary model is a possibility of releasing both p_k and $\bar p_k$ from the Miwa locus, which still remains a challenge in the Hermitian and complex rectangular cases. If resolved, it could extend the Hermitian superintegrability to non-Gaussian cases and to Dijkgraaf-Vafa phases [44].

10. Conclusion

In this paper, we extended the study of superintegrability to unitary matrix models. We showed that it works exactly in the same way as for the Hermitian and rectangular complex models, still some new aspects of the story get revealed by this generalization.

An advantage of unitary models is that the one is in no way restricted to the Gaussian measure, and $\rho(z)$ from (47), while ignored in (9), can easily be made non-trivial by choosing appropriate values of $\bar p_k$. This can be important for further generalizations, say to torus knot model [28,45], where $\rho(z) = \exp(\log^2 z)$, and also the Vandermonde factor in the measure is essentially modified, which needs more work to handle.

The partition function of unitary models can be always expanded into Schur polynomials, without a restriction to Gaussian measures needed so far in the Hermitian case. However, this decomposition includes an explicit dependence on N through the restriction $l_R \le N$ in the sum over partitions R. This restriction can be lifted if the symmetry between U and $U^\dagger$ is broken, and $\bar p_k$ are restricted to a peculiar Miwa locus $\bar p_k = \mathrm{tr}\,\bar X^k$ with $N \times N$ matrix X, thus the strong N-dependence persists. This is exactly in parallel with Hermitian case, still the N dependence now has two different equally efficient descriptions.

Superintegrability can be easily extended from the simplest unitary model to many other more sophisticated examples, see sec. 5–8 above.

Thus the study of unitary models confirms universality of superintegrability, i.e. its applicability to more and more relevant models. At the same time, it raises new questions and can help to better understand this mysteriously general property.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

Throughout the paper, we use the skew Schur functions $S_{R/Q}\{p\}$ [33] defined by

$$S_R\{p+p'\} = \sum_Q S_{R/Q}\{p'\} S_Q\{p\} \tag{49}$$

and the Q Schur functions [33,41], which are defined as the Hall-Littlewood polynomials at the value of parameter $t = -1$:

$$Q_R = \begin{cases} 2^{l_R/2}\cdot \mathrm{HL}_R(t=-1) & \text{for } R \in \mathrm{SP} \\ 0 & \text{otherwise} \end{cases} \tag{50}$$

where SP means strict partitions, i.e. those with all lines of distinct lengths. We fix their normalization as in [18] by the Cauchy identity

$$\sum_R Q_R\{p\} Q_R\{p'\} = \exp\left(\sum_k \frac{p_k p'_k}{k+1/2}\right) \tag{51}$$

At last, we use the shifted Schur functions $S^*_R\{p\}$, which can be unambiguously expressed through the shifted power sums [46]

$$p^*_k := \sum_i \left[(x_i - i)^k - (-i)^k\right] \tag{52}$$

if one requires

$$S^*_\mu\{p^*_k\} = S_\mu\{p^*\} + \sum_{\lambda:\ |\lambda|<|\mu|} c_{\mu\lambda} S_\lambda\{p^*_k\}$$

$$S^*_\mu(R_i) = 0 \qquad \text{if } \mu \notin R \tag{53}$$

Their other definitions and properties can be found in [39].

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