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On the Stone–Čech Compactification Functor and the Normal Extensions of Monoids

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(Submitted by G. G. Amosov)

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Abstract—The paper deals with extensions of algebraic and topological monoids. The extensions are defined by short exact sequences of objects and morphisms in the corresponding categories of monoids and their homomorphisms. We consider the normal extensions of discrete monoids and their Stone–Čech compactifications that are compact right topological monoids. We study properties of the Stone–Čech compactification functor acting from the category of algebraic monoids and their homomorphisms to the category of compact right topological monoids and their continuous homomorphisms. It is shown that this functor preserves the normal extensions of monoids. We also obtain sufficient conditions for preserving the Schreier extensions of monoids under the action of this functor.

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1. INTRODUCTION

The paper is concerned with the normal extensions of monoids. Two categories of monoids are involved in our work. Namely, these are the category of all monoids and their homomorphisms as well as the category of compact right topological monoids and their continuous homomorphisms. We consider the Stone–Čech compactification functor between these categories and investigate its action on the normal extensions of monoids.

The study of extensions is closely connected with many interesting problems in the categories of Algebra and Functional Analysis (see, for example, [1, 2]). A.H. Clifford [3] introduced the ideal extensions in the category of semigroups. L. Rédei [4] defined the Schreier extensions of monoids which are the special case of the normal extensions considered in [5, 6]. It is worth noting that the Schreier extensions of semigroups are intimately related to the extensions of groups.

The motivation for the work presented here comes from our study of the reduced semigroup C^*-algebras in [7, 13] and the normal extensions of semigroups in [14, 16]. In [16], it is shown that the normal extensions of semigroups are applied to solving the functoriality problem for morphisms of semigroup C^*-algebras which was posed, for example, in [17].

As is well known, the binary associative operation m on a discrete monoid (M, m) has a natural extension $\overline{m}$ to the Stone–Čech compactification βM of the discrete topology on M. The pair $(\beta M, \overline{m})$

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is a compact right topological monoid, and M is a subset of its topological center. The investigation of algebraic and topological properties of such compact right topological monoids is very interesting and important. In particular, these monoids have applications to the theory of Banach algebras, to combinatorial number theory and to topological dynamics (see, for example, [18]).

In the present paper we show that the Stone–Čech compactification functor acting from the category of monoids preserves the normal extensions. It means that this functor sends each normal extension of monoids to a normal extension of compact right topological monoids which are the Stone–Čech compactifications of monoids with discrete topologies. Moreover, we obtain conditions under which the Stone–Čech compactification functor preserves the Schreier extensions of monoids.

The paper is organized as follows. It consists of four sections. Section 1 is Introduction. In Section 2 we present some preliminaries and set up notation and terminology. The description of the Stone–Čech compactification functor between the categories of monoids is contained there. Sections 3 deals with the proof of the property for this functor to preserve the normal extensions of monoids. In Section 4 we give conditions under which the Stone–Čech compactification functor preserves the Schreier extensions of monoids.

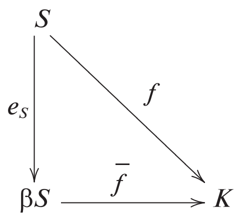
2. BACKGROUND AND DEFINITIONS

We shall use some basic notions and facts from the theory of categories and functors for which a standard reference is Mac Lane's book [19]. For various ways of constructions and for properties of the Stone–Čech compactifications of topological spaces and discrete monoids, we refer the reader, for example, to [20] and [18].

Let us consider *the Stone–Čech compactification functor*

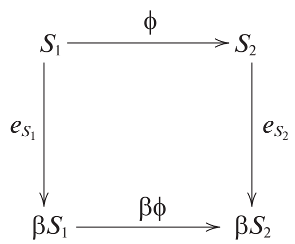
$$\beta : \mathcal{S}et \longrightarrow \mathcal{C}omp\ \mathcal{H}aus$$

from the category $\mathcal{S}et$ of all small sets and all functions between them to the category $\mathcal{C}omp\ \mathcal{H}aus$ of all compact Hausdorff spaces and all continuous functions between them. This functor assigns to each set S the Stone-Čech compactification βS of the discrete topology on S. Recall that for a set S endowed with the discrete topology we have the homeomorphic embedding $e_S : S \longrightarrow \beta S$ such that the image set $im(e_S)$ is dense in the compact space βS and the following universal property is fulfilled. For every compact Hausdorff space K and every function $f : S \longrightarrow K$ there exists a unique continuous function $\overline{f} : \beta S \longrightarrow K$ making the diagram



(1)

commute. Notice that we say that the pair $(\beta S, e_S)$, or simply βS itself, is *the Stone–Čech compactification* of S. The Stone–Čech compactification functor β assigns to every function $\phi : S_1 \longrightarrow S_2$ in $\mathcal{S}et$ the unique continuous function $\beta\phi := \overline{e_{S_2} \circ \phi} : \beta S_1 \longrightarrow \beta S_2$ such that the diagram



(2)

is commutative.

Now we consider the category $\mathcal{M}on$ of all monoids and their morphisms and the category $\mathcal{C}omp\ \mathcal{M}on_r$ consisting of all compact right topological monoids and their continuous morphisms. By morphisms of monoids we mean homomorphisms which preserve identities.

We denote by (M, m) a monoid in the category $\mathcal{M}on$. Here, M is a set and $m : M \times M \longrightarrow M$ is a multiplication in M. As usual, we write xy instead of $m(x, y)$, where $x, y \in M$.

By *a compact right topological monoid* we mean a triple $(M, m, \mathcal{T})$, where (M, m) is a monoid, $\mathcal{T}$ is a compact Hausdorff topology in M, and for each $a \in M$, the right shift mapping

$$\rho_a : M \longrightarrow M : x \longmapsto xa, \quad x \in M,$$

is continuous.

The identities in all algebraic and topological monoids will be denoted by the same symbol 1.

In what follows, we also use the same bold letter $\mathbf{M}$ to denote both objects (M, m) and $(M, m, \mathcal{T})$ from the categories $\mathcal{M}on$ and $\mathcal{C}omp\ \mathcal{M}on_r$ respectively. Thus, we write $\mathbf{M} = (M, m)$ as well as $\mathbf{M} = (M, m, \mathcal{T})$.

The bold Greek letters $\boldsymbol{\mu}, \boldsymbol{\tau}, \boldsymbol{\sigma}, \ldots$ stand for morphisms in both categories $\mathcal{M}on$ and $\mathcal{C}omp\ \mathcal{M}on_r$.

The topological center of a compact right topological monoid $\mathbf{M} = (M, m, \mathcal{T})$ is defined as the subset $\{a \in M \mid \lambda_a \text{ is continuous}\}$ in M, where λ_a is the left shift mapping given by

$$\lambda_a : M \longrightarrow M : x \longmapsto ax, \quad x \in M.$$

Further we shall define a covariant functor

$$\overline{\beta} : \mathcal{M}on \longrightarrow \mathcal{C}omp\ \mathcal{M}on_r.$$

Suppose we are given a monoid $\mathbf{M} = (M, m)$. The binary operation m has a natural extension, denoted by $\overline{m}$, to the Stone–Čech compactification βM of M. This extension is a unique binary operation on βM such that for each $a \in \beta M$ the right shift mapping $\rho_a : \beta M \longrightarrow \beta M$ is continuous as well as for each $b \in M$ the left shift mapping $\lambda_b : \beta M \longrightarrow \beta M$ is continuous [18, Theorem 4.1]. Under the operation $\overline{m}$, the compactification $(\beta M, \overline{m}, \mathcal{T})$ is a compact right topological monoid with M contained in its topological center [18, Theorems 4.1, 4.4]. As usual, we identify M with the image set $im(e_M)$ in βM. The identity of $\mathbf{M}$ is also the identity for the compact right topological monoid $(\beta M, \overline{m}, \mathcal{T})$ (see [18, Subsection 9.4])).

By definition, we put that $\overline{\beta}$ assigns to an object $\mathbf{M} = (M, m)$ of the category $\mathcal{M}on$ the object $(\beta M, \overline{m}, \mathcal{T})$ of the category $\mathcal{C}omp\ \mathcal{M}on_r$, that is, we write $\overline{\beta}\mathbf{M} = (\beta M, \overline{m}, \mathcal{T})$.

Before defining the action of $\overline{\beta}$ on morphisms, we introduce two forgetful functors

$$U_1 : \mathcal{M}on \longrightarrow \mathcal{S}et \quad \text{and} \quad U_2 : \mathcal{C}omp\ \mathcal{M}on_r \longrightarrow \mathcal{C}omp\ \mathcal{H}aus.$$

The former assigns to each monoid $\mathbf{M} = (M, m)$ its underlying set M and to each morphism $\boldsymbol{\mu} : \mathbf{M} \longrightarrow \mathbf{N}$ of monoids the same function regarded just as a function between the sets M and N and denoted by $\mu : M \longrightarrow N$. Thus, we shall write $U_1\mathbf{M} = M$ and $U_1\boldsymbol{\mu} = \mu$.

The latter assigns to each compact right topological monoid $\mathbf{M} = (M, m_1, \mathcal{T}_1)$ its underlying topological space $(M, \mathcal{T}_1)$ and to each morphism $\boldsymbol{\mu} : \mathbf{M} \longrightarrow \mathbf{N} = (N, m_2, \mathcal{T}_2)$ from $\mathcal{C}omp\ \mathcal{M}on_r$ the same function regarded just as a continuous function between the topological spaces $(M, \mathcal{T}_1)$ and $(N, \mathcal{T}_2)$. We shall use the notation $U_2\mathbf{M} = M$ and $U_2\boldsymbol{\mu} = \mu : M \longrightarrow N$.

To define the action of $\overline{\beta}$ on morphisms we take an arbitrary arrow

$$\boldsymbol{\phi} : \mathbf{S_1} = (S_1, m_1) \longrightarrow \mathbf{S_2} = (S_2, m_2) \tag{3}$$

in the category $\mathcal{M}on$. Then we consider the morphism $\beta U_1\boldsymbol{\phi} = \beta\phi : \beta S_1 \longrightarrow \beta S_2$ in the category $\mathcal{C}omp\ \mathcal{H}aus$.

By [18, Theorem 4.1 (c)], the image set $im(e_{S_2})$ is contained in the topological center of the compact right topological monoid $(\beta S_2, \overline{m}_2, \mathcal{T}_2)$. Hence, the image set $im(e_{S_2} \circ \phi)$ is a subset of the topological center of the monoid $(\beta S_2, \overline{m}_2, \mathcal{T}_2)$. By [18, Theorem 4.8], this property together with the fact that $\beta\phi$ is the unique continuous function making diagram (2) commute guarantee that $\beta\phi$ is a homomorphism. In addition, the commutativity of diagram (2) yields the equalities

$$\beta\phi(1) = \beta\phi(e_{S_1}(1)) = e_{S_2}(\phi(1)) = e_{S_2}(1) = 1,$$

that is, $\beta\phi$ preserves the identities.

Thus, for morphism (3) in the category $\mathcal{M}on$, one has the morphism

$$\overline{\beta}\boldsymbol{\phi} : (\beta S_1, \overline{m}_1, \mathcal{T}_1) \longrightarrow (\beta S_2, \overline{m}_2, \mathcal{T}_2)$$

in the category $\mathcal{C}omp\ \mathcal{M}on_r$ which is defined to be as a continuous function just $\beta\phi$. Note, we have

$$U_2\overline{\beta}\boldsymbol{\phi} = \beta\phi = \beta U_1\boldsymbol{\phi}.$$

It is straightforward to check that for every identity morphism $\mathbf{1_M} : \mathbf{M} \longrightarrow \mathbf{M}$ and for arbitrary morphisms $\boldsymbol{\tau} : \mathbf{S} \longrightarrow \mathbf{L}$ and $\boldsymbol{\sigma} : \mathbf{L} \longrightarrow \mathbf{M}$ in the category $\mathcal{M}on$ the following properties are fulfilled:

$$\overline{\beta}\mathbf{1_M} = \mathbf{1}_{\overline{\beta}\mathbf{M}} \quad \text{and} \quad \overline{\beta}(\boldsymbol{\sigma} \circ \boldsymbol{\tau}) = \overline{\beta}\boldsymbol{\sigma} \circ \overline{\beta}\boldsymbol{\tau}.$$

Therefore we conclude that $\overline{\beta}$ is a covariant functor. We call $\overline{\beta}$ *the Stone–Čech compactification functor* between the categories of monoids.

Using the preceding observations, it is easily seen that the diagram

$$\begin{array}{ccc} \mathcal{M}on & \xrightarrow{\ \overline{\beta}\ } & \mathcal{C}omp\ \mathcal{M}on_r \\ \Big\downarrow{\scriptstyle U_1} & & \Big\downarrow{\scriptstyle U_2} \\ \mathcal{S}et & \xrightarrow{\ \beta\ } & \mathcal{C}omp\ \mathcal{H}aus \end{array} \tag{4}$$

commutes.

Further we recall definitions concerning extensions of monoids. For details in the theory of semigroup extensions we refer the reader to [5, 6, 21, 22].

Let $\mathbf{S}$, $\mathbf{L}$, $\mathbf{M}$ and $\boldsymbol{\tau} : \mathbf{S} \longrightarrow \mathbf{L}$, $\boldsymbol{\sigma} : \mathbf{L} \longrightarrow \mathbf{M}$ be objects and morphisms either in the category $\mathcal{M}on$ or in the category $\mathcal{C}omp\ \mathcal{M}on_r$. The sequence

$$\mathbf{S} \xrightarrow{\ \boldsymbol{\tau}\ } \mathbf{L} \xrightarrow{\ \boldsymbol{\sigma}\ } \mathbf{M} \tag{5}$$

is said to be *exact at the term* $\mathbf{L}$ if the image set $\tau(S)$ coincides with the preimage set for the identity of $\mathbf{M}$ under the function σ, that is, the equality $\sigma^{-1}(1) = \tau(S)$ holds, where $S = U_1\mathbf{S}$. Note that for $z \in M$ we write $\sigma^{-1}(z)$ instead of $\sigma^{-1}(\{z\})$.

The triple $(\mathbf{L}, \boldsymbol{\tau}, \boldsymbol{\sigma})$ is called *a normal extension* of the monoid $\mathbf{S}$ by means of the monoid $\mathbf{M}$ if $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ are injective and surjective respectively, and sequence (5) is exact at the term $\mathbf{L}$.

In other words, every normal extension is defined by a short exact sequence

$$\mathbf{1} \longrightarrow \mathbf{S} \xrightarrow{\ \boldsymbol{\tau}\ } \mathbf{L} \xrightarrow{\ \boldsymbol{\sigma}\ } \mathbf{M} \longrightarrow \mathbf{1} \tag{6}$$

consisting of monoids and their homomorphisms. Throughout, the symbol $\mathbf{1}$ stands for the trivial monoids in the categories $\mathcal{M}on$ and $\mathcal{C}omp\ \mathcal{M}on_r$. It is worth noting, that we use here the terminology of [22]. Namely, a short sequence of monoids (6) is said to be *exact* if $\boldsymbol{\tau}$ is injective, $\boldsymbol{\sigma}$ is surjective and sequence (6) is exact at the term $\mathbf{L}$.

A triple $(\mathbf{L}, \boldsymbol{\tau}, \boldsymbol{\sigma})$ is called *a Schreier extension* of the monoid $\mathbf{S}$ by means of the monoid $\mathbf{M}$ if this triple is a normal extension and there exists a subset

$$\{x_z \mid \sigma(x_z) = z,\ z \in M \setminus \{1\}\} \tag{7}$$

in L having the following property: every element $y \in L \setminus \tau(S)$ has a unique factorization in the form $y = x_z\tau(a)$, where $z \in M \setminus \{1\}$ and $a \in S$. In that case, the set L can be decomposed as follows:

$$L = \tau(S) \bigsqcup \left(\bigsqcup_{z \in M \setminus \{1\}} x_z\tau(S) \right), \tag{8}$$

where we set $x_z\tau(S) := \{x_z\tau(a) \mid a \in S\}$.

Often we say that the monoid $\mathbf{L}$ itself is an extension of the monoid $\mathbf{S}$ by the monoid $\mathbf{M}$. Note that in this case some authors say that $\mathbf{L}$ is an extension of the monoid $\mathbf{M}$ by the monoid $\mathbf{S}$.

3. NORMAL EXTENSIONS OF MONOIDS AND THEIR STONE–ČECH COMPACTIFICATIONS

In this section we show that the Stone–Čech compactification functor $\overline{\beta}$ between the categories $\mathcal{M}on$ and $\mathcal{C}omp\ \mathcal{M}on_r$ preserves normal extensions of monoids. We divide the proof of this statement into two lemmas.

As is known, the Stone–Čech compactification functor $\beta : \mathcal{S}et \longrightarrow \mathcal{C}omp\ \mathcal{H}aus$ preserves injective and surjective morphisms (see, for example [18, Exercise 3.4.1]). Using the commutativity of diagram (4) and the property of any functor to preserve the retractions and the coretractions, one has the first lemma stating that the Stone–Čech compactification functor between the categories of monoids preserves injective and surjective morphisms as well. For the sake of completeness, we prove this lemma.

Lemma 1. *Let* $\boldsymbol{\mu} : \mathbf{M} \longrightarrow \mathbf{N}$ *be a morphism in the category* $\mathcal{M}on$. *Then the morphism* $\overline{\beta}\boldsymbol{\mu} : \overline{\beta}\mathbf{M} \longrightarrow \overline{\beta}\mathbf{N}$ *is injective if* $\boldsymbol{\mu}$ *is injective, and surjective if* $\boldsymbol{\mu}$ *is surjective.*

Proof. Let us assume that a morphism $\boldsymbol{\mu} : \mathbf{M} \longrightarrow \mathbf{N}$ is an injection. It follows that $\mu : M \longrightarrow N$ is an injective function. Since a morphism of the category $\mathcal{S}et$ is injective if and only if it is a coretraction, the function μ is a coretraction. But every covariant functor preserves coretractions, so that the morphism $\beta\mu$ has a left inverse morphism in the category $\mathcal{C}omp\ \mathcal{H}aus$. Hence, $\beta\mu$ is injective.

The commutativity of diagram (4) implies the equality $U_2\overline{\beta}\boldsymbol{\mu} = \beta\mu$. As a consequence, the morphisms $U_2\overline{\beta}\boldsymbol{\mu}$ as well as $\overline{\beta}\boldsymbol{\mu}$ are injective. In the similar way one shows that the Stone–Čech compactification functor $\overline{\beta}$ preserves surjections. □

Adjointness of the Stone–Čech compactification functor β is involved in the proof of the following lemma. Before giving this proof, we recall some standard facts about the adjoint functors and the coproducts in the categories $\mathcal{S}et$ and $\mathcal{C}omp\ \mathcal{H}aus$.

The Stone–Čech compactification functor β is left adjoint to the forgetful functor (see [19, p. 125])

$$V : \mathcal{C}omp\ \mathcal{H}aus \longrightarrow \mathcal{S}et$$

which sends each compact Hausdorff space to the underlying set of its points and forgets about the continuity of functions.

As is known, one of the most useful properties of the left adjoint functors is to preserve the colomits, in particular, the coproducts which exist in domains of those functors [19, Ch. V, § 5].

For arbitrary objects A_1 and A_2 in $\mathcal{S}et$, a disjoint union $A_1 \bigsqcup A_2$ together with the natural embeddings

$$A_1 \xrightarrow{\ i_1\ } A_1 \bigsqcup A_2 \xleftarrow{\ i_2\ } A_2.$$

is a coproduct of A_1 and A_2. Since the functor β preserves the coproducts, one has the coproduct diagram

$$\beta A_1 \xrightarrow{\ \beta i_1\ } \beta\left(A_1 \bigsqcup A_2\right) \xleftarrow{\ \beta i_2\ } \beta A_2$$

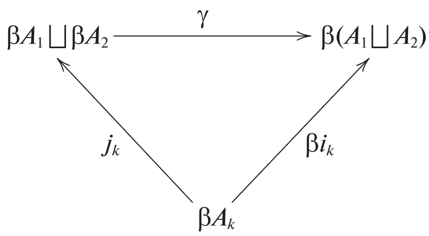
in the category $\mathcal{C}omp\ \mathcal{H}aus$.

On the other hand, a topological sum $\beta A_1 \bigsqcup \beta A_2$ with the natural embeddings

$$\beta A_1 \xrightarrow{\ j_1\ } \beta A_1 \bigsqcup \beta A_2 \xleftarrow{\ j_2\ } \beta A_2$$

is a coproduct of objects βA_1 and βA_2 in the category $\mathcal{C}omp\ \mathcal{H}aus$.

It follows from the universal property for coproducts that there exists a unique isomorphism $\gamma : \beta A_1 \bigsqcup \beta A_2 \longrightarrow \beta(A_1 \bigsqcup A_2)$ in the category $\mathcal{C}omp\ \mathcal{H}aus$ such that the diagram



is commutative for each $k = 1, 2$. As a consequence, one has the equality

$$\beta\left(A_1 \bigsqcup A_2\right) = \beta i_1(\beta A_1) \bigsqcup \beta i_2(\beta A_2). \tag{9}$$

In the sequel, we shall make use of equality (9) for proving Lemma 2 and Theorem 2.

Lemma 2. *Let a sequence of objects and morphisms in the category $\mathcal{M}on$*

$$\mathbf{S} \xrightarrow{\ \boldsymbol{\tau}\ } \mathbf{L} \xrightarrow{\ \boldsymbol{\sigma}\ } \mathbf{M} \tag{10}$$

be exact at the term $\mathbf{L}$. *Then the sequence of objects and morphisms in the category $\mathcal{C}omp\ \mathcal{M}on_r$,*

$$\overline{\beta}\mathbf{S} \xrightarrow{\ \overline{\beta}\boldsymbol{\tau}\ } \overline{\beta}\mathbf{L} \xrightarrow{\ \overline{\beta}\boldsymbol{\sigma}\ } \overline{\beta}\mathbf{M} \tag{11}$$

is exact at the term $\overline{\beta}\mathbf{L}$.

Proof. The exactness of sequences (10) and (11) at the middle terms is equivalent to the validity of the equalities $im(\tau) = \sigma^{-1}(1)$ and

$$im(\beta\tau) = (\beta\sigma)^{-1}(1) \tag{12}$$

in the categories $\mathcal{S}et$ and $\mathcal{C}omp\ \mathcal{H}aus$ respectively. To prove that equality (12) holds we shall consider two decompositions of the compact space βL.

Firstly, we consider the disjoint union $L = \sigma^{-1}(1) \bigsqcup (L \backslash \sigma^{-1}(1))$ and the natural embeddings

$$l_1 : \sigma^{-1}(1) \longrightarrow L \quad \text{and} \quad l_2 : L \backslash \sigma^{-1}(1) \longrightarrow L$$

in the category $\mathcal{S}et$.

Since the left adjoint functor $\beta : \mathcal{S}et \longrightarrow \mathcal{C}omp\ \mathcal{H}aus$ preserves the coproducts we get (see (9))

$$\beta L = \beta\left(\sigma^{-1}(1) \bigsqcup (L \backslash \sigma^{-1}(1))\right) = \beta l_1(\beta\sigma^{-1}(1)) \bigsqcup \beta l_2(\beta(L \backslash \sigma^{-1}(1))). \tag{13}$$

Now, we show that the equality

$$\beta l_1(\beta\sigma^{-1}(1)) = im(\beta\tau) \tag{14}$$

holds. To this aim, let us consider the function $\tau^{im} : S \longrightarrow im(\tau) = \sigma^{-1}(1)$ which is the corestriction of the function τ to the set $im(\tau)$. Obviously, one has the equality $\tau = l_1 \circ \tau^{im}$. Because the function τ^{im} is surjective, the function $\beta\tau^{im}$ is also surjective [18, Exercise 3.4.1]. Hence, we get equality (14) as follows:

$$im(\beta\tau) = im(\beta l_1 \circ \beta\tau^{im}) = im(\beta l_1) = \beta l_1(\beta\sigma^{-1}(1)).$$

Therefore, combining (13) with (14), we have the first decomposition of the space βL:

$$\beta L = im(\beta\tau) \bigsqcup \beta l_2(\beta(L \backslash \sigma^{-1}(1))). \tag{15}$$

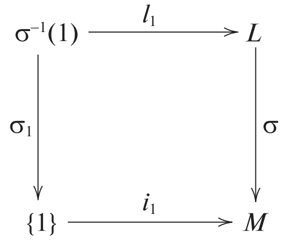
Furthermore, let us consider the second decomposition of the space βL:

$$\beta L = (\beta\sigma)^{-1}(1) \bigsqcup (\beta\sigma)^{-1}(\beta M \setminus \{1\}). \tag{16}$$

We prove that one has the following relations:

$$im(\beta\tau) \subset (\beta\sigma)^{-1}(1) \quad \text{and} \quad \beta l_2(\beta(L \backslash \sigma^{-1}(1))) \subset (\beta\sigma)^{-1}(\beta M \setminus \{1\}). \tag{17}$$

To prove the former, let us consider the mapping $\sigma_1 : \sigma^{-1}(1) \longrightarrow \{1\}$ and the natural embedding $i_1 : \{1\} \longrightarrow M$ in the category $\mathcal{S}et$. Obviously, the diagram



is commutative, that is, $\sigma \circ l_1 = i_1 \circ \sigma_1$ which implies the relation $\beta\sigma \circ \beta l_1 = \beta i_1 \circ \beta\sigma_1$. By (14), one has the following equalities:

$$im(\beta\sigma \circ \beta l_1) = \beta\sigma(\beta l_1(\beta\sigma^{-1}(1))) = \beta\sigma(im(\beta\tau)), \tag{18}$$

Because the functions σ_1 and $\beta\sigma_1$ are surjective, and βi_1 is unit preserving, we get the equalities

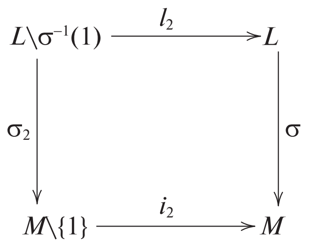
$$im(\beta i_1 \circ \beta\sigma_1) = im(\beta i_1) = \{1\}. \tag{19}$$

It follows from relations (18) and (19) that in (17) the first inclusion is valid.

To prove that in (17) the second incluson holds, we proceed as follows. In the category $\mathcal{S}et$ let us take the mapping σ_2 which is defined by

$$\sigma_2 : L \setminus \sigma^{-1}(1) \longrightarrow M \setminus \{1\} : x \longmapsto \sigma(x),$$

and the natural embedding $i_2 : M \setminus \{1\} \longrightarrow M$. Consider the commutative diagram



Then we have the equality

$$\beta\sigma \circ \beta l_2 = \beta i_2 \circ \beta\sigma_2. \tag{20}$$

for the compositions of the mappings in the category Comp Haus. Note that one has

$$im(\beta\sigma \circ \beta l_2) = \beta\sigma(\beta l_2(\beta(L \setminus \sigma^{-1}(1))); \tag{21}$$

$$im(\beta i_2 \circ \beta\sigma_2) \subset im(\beta i_2) = \beta i_2(\beta(M \setminus \{1\})). \tag{22}$$

Hence, relations (20), (21) and (22) imply

$$\beta\sigma(\beta l_2(\beta(L \setminus \sigma^{-1}(1))) \subset im(\beta i_2) = \beta i_2(\beta(M \setminus \{1\})). \tag{23}$$

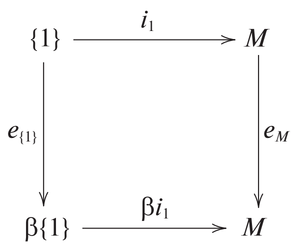
We claim that

$$\beta i_2(\beta(M \setminus \{1\})) = \beta M \setminus \{1\}. \tag{24}$$

Indeed, since the left adjoint functor β preserves the coproducts we have the decomposition (compare with (9))

$$\beta M = \beta i_2(\beta(M \setminus \{1\})) \bigsqcup \beta i_1(\beta\{1\}). \tag{25}$$

It follows from the commutativity of the diagram



that the following equalities hold:

$$\beta i_1(\beta\{1\}) = \beta i_1(e_{\{1\}}(\{1\})) = e_M(i_1(\{1\})) = e_M(\{1\}) = \{1\}. \tag{26}$$

Combining (25) and (26), we obtain equality (24), as claimed. Therefore, relations (23) and (24) yield the second inclusion in (17).

Finally, using (15), (16) and (17), we get equality (12), as desired. □

As an immediate consequence of Lemma 1 and Lemma 2, we obtain

Theorem 1. *Let* $\overline{\beta} : \mathcal{M}on \longrightarrow \mathcal{C}omp\ \mathcal{M}on_r$ *be the Stone–Čech compactification functor. Let a triple* $(\mathbf{L}, \boldsymbol{\tau}, \boldsymbol{\sigma})$ *be a normal extension of a monoid* $\mathbf{S}$ *by a monoid* $\mathbf{M}$. *Then the triple* $(\overline{\beta}\mathbf{L}, \overline{\beta}\boldsymbol{\tau}, \overline{\beta}\boldsymbol{\sigma})$ *is a normal extension of the compact right topological monoid* $\overline{\beta}\mathbf{S}$ *by the compact right topological monoid* $\overline{\beta}\mathbf{M}$.

4. SCHREIER EXTENSIONS OF MONOIDS AND THEIR STONE–ČECH COMPACTIFICATIONS

This section is concerned with the Schreier extensions of monoids by means of finite monoids.

Firstly, for a such extension $(\mathbf{L}, \boldsymbol{\tau}, \boldsymbol{\sigma})$ we prove Theorem 2 on a decomposition of the set space βL. Secondly, Theorem 3 states that the Stone–Čech compactification functor $\overline{\beta}$ transfers a Schreier extension of monoids $(\mathbf{L}, \boldsymbol{\tau}, \boldsymbol{\sigma})$, where $\mathbf{L}$ has the left cancellation property, to the Schreier extension of the Stone–Čech compactifications of the compact right topological monoids.

Let us consider a normal extension $(\mathbf{L}, \boldsymbol{\tau}, \boldsymbol{\sigma})$ of monoids defined by a short exact sequence

$$\mathbf{1} \longrightarrow \mathbf{S} \xrightarrow{\ \boldsymbol{\tau}\ } \mathbf{L} \xrightarrow{\ \boldsymbol{\sigma}\ } \mathbf{F} \longrightarrow \mathbf{1} \tag{27}$$

in the category $\mathcal{M}on$, where $\mathbf{F}$ is a finite monoid. Since the finite set F, which is equipped with the discrete topology, is a compact Hausdorff space, the pair $(F, 1_F)$ is its Stone–Čech compactification. Of course, the multiplication $\overline{m}$ in the Stone–Čech compactification $\mathbf{F}$ coincides with the multiplication m in $\mathbf{F}$. Therefore, in that case, after applying the Stone–Čech compactification functor $\overline{\beta}$ to sequence (27), we can consider the short exact sequence

$$\mathbf{1} \longrightarrow \overline{\beta}\mathbf{S} \xrightarrow{\ \overline{\beta}\boldsymbol{\tau}\ } \overline{\beta}\mathbf{L} \xrightarrow{\ \overline{\beta}\boldsymbol{\sigma}\ } \mathbf{F} \longrightarrow \mathbf{1},$$

and say that it defines the normal extension $(\overline{\beta}\mathbf{L}, \overline{\beta}\boldsymbol{\tau}, \overline{\beta}\boldsymbol{\sigma})$ of the compact right topological monoid $\overline{\beta}\mathbf{S}$ by $\mathbf{F}$.

Theorem 2. *Let* $\overline{\beta} : \mathcal{M}on \longrightarrow \mathcal{C}omp\ \mathcal{M}on_r$ *be the Stone–Čech compactification functor. Let a triple* $(\mathbf{L}, \boldsymbol{\tau}, \boldsymbol{\sigma})$ *be a Schreier extension of a monoid* $\mathbf{S}$ *by a finite monoid* $\mathbf{F}$. *Then* $(\overline{\beta}\mathbf{L}, \overline{\beta}\boldsymbol{\tau}, \overline{\beta}\boldsymbol{\sigma})$ *is a normal extension of the compact right topological monoid* $\overline{\beta}\mathbf{S}$ *by* $\mathbf{F}$. *Moreover, there is a subset*

$$\{\overline{x}_z \mid \beta\sigma(\overline{x}_z) = z,\ z \in F \setminus \{1\}\} \tag{28}$$

in the set βL *such that one has the decomposition*

$$\beta L = \beta\tau(\beta S) \bigsqcup \left(\bigsqcup_{z \in F\setminus\{1\}} \overline{x}_z \beta\tau(\beta S) \right). \tag{29}$$

Proof. Since the triple $(\mathbf{L}, \boldsymbol{\tau}, \boldsymbol{\sigma})$ is a Schreier extension of monoids, we have finite set (7) and finite decomposition (8) with $z \in F \setminus \{1\}$ instead of $z \in M \setminus \{1\}$.

Consider the natural embeddings in the category $\mathcal{S}et$

$$j :\ \tau(S) \longrightarrow L \quad \text{and} \quad j_z : x_z\tau(S) \longrightarrow L$$

for every $z \in F \setminus \{1\}$.

Because the Stone–Čech compactification functor preserves the coproducts, we obtain (see (9))

$$\beta L = \beta j(\beta(\tau(S))) \bigsqcup \left(\bigsqcup_{z \in F\setminus\{1\}} \beta j_z(\beta(x_z\tau(S))) \right). \tag{30}$$

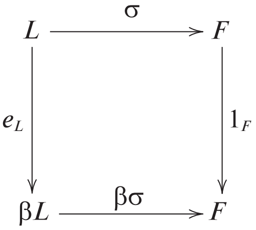
The arguments similar to those in the proof of Lemma 2 show that we have

$$\beta j(\beta(\tau(S))) = \beta\tau(\beta S). \tag{31}$$

Combining (30) and (31), we get the decomposition

$$\beta L = \beta\tau(\beta S) \bigsqcup \left(\bigsqcup_{z \in F \setminus \{1\}} \beta j_z(\beta(x_z \tau(S))) \right). \tag{32}$$

It follows from the commutativity of the diagram

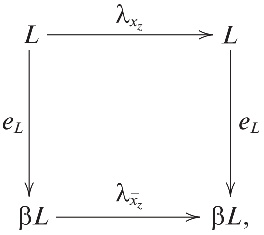


that the value of the function $\beta\sigma : \beta L \longrightarrow F$ at the point $\overline{x}_z := e_L(x_z) \in \beta L$ is equal to the element $z \in F \setminus \{1\}$.

Further, we claim that, for each $z \in F \setminus \{1\}$, the equality

$$\overline{x}_z \beta\tau(\beta S) = \beta j_z(\beta(x_z \tau(S))) \tag{33}$$

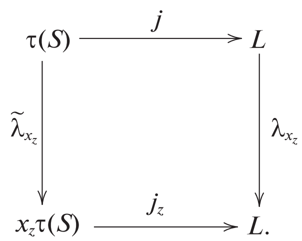
holds. Indeed, let us consider the left shift mappings $\lambda_{x_z} : M \longrightarrow M$ and $\lambda_{\overline{x}_z} : \beta L \longrightarrow \beta L$. As was noted in Section 2, the element $\overline{x}_z$ belongs to the topological center of the compact right topological monoid $\overline{\beta}\mathbf{L}$, that is, the function $\lambda_{\overline{x}_z}$ is continuous. Hence, the commutativity of the diagram



the universal property of the Stone–Čech compactification βL (see (2)) and the definition of the function $\beta\lambda_{x_z}$ (see (2)) yield the equality of continuous functions

$$\lambda_{\overline{x}_z} = \beta\lambda_{x_z}. \tag{34}$$

Consider the left shift mapping $\tilde{\lambda}_{x_z} : \tau(S) \longrightarrow x_z\tau(S) : \tau(a) \longrightarrow x_z\tau(a)$, where $a \in S$. Notice, it is surjective. Hence, the function $\beta\tilde{\lambda}_{x_z} : \beta(\tau(S)) \longrightarrow \beta(x_z\tau(S))$ is also surjective. Furthermore, we have the commutative diagram



Therefore, applying the Stone–Čech compactification functor β to the equality $\lambda_{x_z} \circ j = j_z \circ \tilde{\lambda}_{x_z}$ of compositions, we get

$$\beta\lambda_{x_z} \circ \beta j = \beta j_z \circ \beta\tilde{\lambda}_{x_z}. \tag{35}$$

We are now in a position to show the validity of equality (33). According to the above arguments, one has the following:

$$\overline{x}_z \beta\tau(\beta S) = \overline{x}_z(\beta j \beta(\tau(S))) \quad \text{(by (31))}$$

$$= (\beta\lambda_{x_z} \circ \beta j)(\beta(\tau(S))) \quad \text{(by (34))}$$
$$= (\beta j_z \circ \beta\tilde{\lambda}_{x_z})(\beta(\tau(S))) \quad \text{(by (35))}$$
$$= \beta j_z(\beta(x_z\tau(S))) \quad \text{(by surjectivity of } \beta\tilde{\lambda}_{x_z}),$$

as claimed.

Using (32) and (33), we obtain equality (29), as required. □

Remark 1. In general, the extension of compact right topological monoids constructed in Theorem 2 is not a Schreier extension. Really, for an element $y \in \beta L \setminus \beta\tau(\beta S)$ its factorization in the form $y = \overline{x}_z\beta\tau(a)$, where $z \in F \setminus \{1\}$ and $a \in \beta S$, may not be unique.

The following theorem gives a sufficient condition for an extension of the Stone–Čech compactification of a compact right topological monoid to be a Schreier extension.

Before formulating this result, we recall that an element a in a monoid $\mathbf{M}$ is said to be *left cancelable* if whenever $x, y \in \mathbf{M}$ and $ax = ay$, one has $x = y$. In other words, the left shift mapping $\lambda_a : M \to M$ is injective. A monoid $\mathbf{M}$ is said to be *left cancellative*, or *with the left cancellation property*, if every $a \in \mathbf{M}$ is left cancelable.

Theorem 3. *Let $\overline{\beta} : \mathcal{M}on \longrightarrow \mathcal{C}omp\,\mathcal{M}on_r$ be the Stone–Čech compactification functor. Let $(\mathbf{L}, \boldsymbol{\tau}, \boldsymbol{\sigma})$ be a Schreier extension of a monoid $\mathbf{S}$ by a finite monoid $\mathbf{F}$ and $\mathbf{L}$ be a left cancellative monoid. Then $(\overline{\beta}\mathbf{L}, \overline{\beta}\boldsymbol{\tau}, \overline{\beta}\boldsymbol{\sigma})$ is a Schreier extension of the compact right topological monoid $\overline{\beta}\mathbf{S}$ by $\mathbf{F}$.*

Proof. By Theorem 2, the triple $(\overline{\beta}\mathbf{L}, \overline{\beta}\boldsymbol{\tau}, \overline{\beta}\boldsymbol{\sigma})$ is a normal extension. Moreover, we are given decomposition (29) for the set βL constructed by making use of decomposition (8) with $z \in F \setminus \{1\}$ instead of $z \in M \setminus \{1\}$. Let us take an arbitrary element $y \in \beta L \setminus \beta\tau(\beta S)$. Then there exists a unique element $\overline{x}_z$ in set (28) such that $y \in \overline{x}_z\beta\tau(\beta S)$.

Further, we take the element $x_z \in L$ from set (7). Notice that $\overline{x}_z = e_L(x_z)$. Since the monoid $\mathbf{L}$ is left cancellative, the element x_z is left cancelable. By [18, Lemma 8.1], the element $\overline{x}_z$ is also left cancelable.

As a consequence, because the function $\beta\tau$ is injective, there exists a unique element $a \in \beta S$ such that $y = \overline{x}_z\beta\tau(a)$, as required. □

Remark 2. Let a triple $(\mathbf{L}, \boldsymbol{\tau}, \boldsymbol{\sigma})$ be a Schreier extension of a monoid $\mathbf{S}$ by a finite monoid $\mathbf{F}$. From the proof of Theorem 3, one sees that $(\overline{\beta}\mathbf{L}, \overline{\beta}\boldsymbol{\tau}, \overline{\beta}\boldsymbol{\sigma})$ is a Schreier extension if there exists set (7) in L such that each element x_z is left cancelable.

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