ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2021, Vol. 42, No. 10, pp. 2386–2391. © Pleiades Publishing, Ltd., 2021.

A Semigroup C^*-Algebra Which Is a Free Banach Module

E. V. Lipacheva[*]

(Submitted by G. G. Amosov)

Chair of Higher Mathematics, Kazan State Power Engineering University, Kazan, 420066 Russia

Received April 29, 2021; revised May 12, 2021; accepted May 18, 2021

Abstract—We consider the reduced semigroup C^*-algebras for monoids with the cancellation property. If there exists a surjective semigroup homomorphism from a monoid onto a group then the corresponding semigroup C^*-algebra can be endowed with the structure of a Banach module over its C^*-subalgebra. For a such monoid, we give conditions under which this Banach module is free.

DOI: 10.1134/S1995080221100152

Keywords and phrases: *reduced semigroup C^*-algebra, free Banach module, graded C^*-algebra, cyclic Banach module, topological isomorphism of Banach modules.*

INTRODUCTION

The note is concerned with the reduced semigroup C^*-algebras which are generated by the left regular representations of semigroups with the cancellation property. These algebras are studied by Coburn [1, 2], Douglas [3] and Murphy [4, 5]. Further, the theory of semigroup C^*-algebras was developed in the papers by a number of authors (see, for example, [6] and references therein).

We studied properties of the reduced semigroup C^*-algebras in [7–17]. The work presented here is a continuation of the research carried out in [18]. There we constructed a topological grading of a semigroup C^*-algebra $C_r^*(S)$ by means of an arbitrary group G. Moreover, the C^*-algebra $C_r^*(S)$ was endowed with the structure of a left Banach module over its C^*-subalgebra $\mathfrak{A}_e$, where e is the unit of the group G. In the case of a finite group G, it was proved that $C_r^*(S)$ is a finitely generated projective Hilbert $\mathfrak{A}_e$-module.

In this note we give conditions under which the $\mathfrak{A}_e$-module $C_r^*(S)$ is a free Banach module. The grading of the C^*-algebra $C_r^*(S)$ is involved in the proof of the main result on a free Banach module. As is known, a grading of an object in a category allows to understand better the structure of this object. In the category of C^*-algebras, one deals with the gradings which are also called the C^*-bundles, or the Fell bundles. Recall that the notion of the topologically graded C^*-algebra was introduced by Excel (see for example [19]) with the aim to define non-commutative versions for concepts of harmonic analysis.

The note consists of Introduction and two Sections. Section 1 contains the necessary information about the semigroup C^*-algebras and the Banach modules over C^*-algebras. In Section 2 we formulate and prove the results on free Banach $\mathfrak{A}_e$-modules.

[*]E-mail: elipacheva@gmail.com

1. PRELIMINARIES

Throughout the note S stands for a discrete cancellative monoid with the identity e.

The main object of our study is the reduced semigroup C^*-algebra $C_r^*(S)$ which is defined as follows.

Let us consider the Hilbert space of all square summable complex-valued functions defined on the monoid S:

$$l^2(S) := \{f : S \to \mathbb{C} \mid \sum_{a\in S} |f(a)|^2 < +\infty\}.$$

The canonical orthonormal basis in the Hilbert space $l^2(S)$ is denoted by $\{e_a \mid a \in S\}$, where

$$e_a(b) := \begin{cases} 1, & \text{if } a = b\,; \\ 0, & \text{if } a \neq b\,. \end{cases}$$

The reduced semigroup C^*-algebra $C_r^*(S)$ is the C^*-subalgebra generated by the set of isometries $\{T_a \mid a \in S\}$ in the algebra of all bounded operators on $l^2(S)$, where the operator T_a is given by the formula

$$T_a(e_b) = e_{ab},\ a, b \in S.$$

Further, we recall the necessary definitions concerning modules. Notice, by a module we mean a left module over an algebra. For more information about the Banach modules, the reader is referred to the book [20].

Let $\mathfrak{A}$ be a unital Banach algebra. A module $\mathfrak{M}$ over the algebra $\mathfrak{A}$ is called *a Banach $\mathfrak{A}$-module* if $\mathfrak{M}$ is a Banach space with a norm satisfying the inequality $||A \cdot M|| \le ||A||\ ||M||$ for all $A \in \mathfrak{A}, M \in \mathfrak{M}$. A subset X of the Banach $\mathfrak{A}$-module $\mathfrak{M}$ is called *a generating set* if the set of all finite $\mathfrak{A}$-linear combinations of elements from X is dense in $\mathfrak{M}$.

An element M of an $\mathfrak{A}$-module $\mathfrak{M}$ is said to be *cyclic* if the equality

$$\mathfrak{M} = \mathfrak{A} \cdot M := \{A \cdot M \mid A \in \mathfrak{A}\}$$

holds. A module having a cyclic element is itself called *a cyclic (or one-generator) module*. We recall that a Banach $\mathfrak{A}$-module $\mathfrak{M}$ is cyclic if and only if it is isomorphic to the quotient module $\mathfrak{A}/I$ for a closed left modular ideal I. Moreover, to construct a such isomorphism the annihilator of $\mathfrak{M}$, which is the kernel of the representation associated with $\mathfrak{M}$, can be taken as the ideal I in the algebra $\mathfrak{A}$ [20, Proposition (VI.2.3)].

Let E be a Banach space. There is the structure of a unital left Banach $\mathfrak{A}$-module in the projective tensor product $\mathfrak{A}\hat{\otimes}E$ which is uniquely determined by the formula

$$A \cdot (B \otimes X) = AB \otimes X, \quad A, B \in \mathfrak{A}, X \in E.$$

A module is called a *free* unital Banach $\mathfrak{A}$-module if it is topologically isomorphic to the module $\mathfrak{A}\hat{\otimes}E$ for some Banach space E. In particular, the algebra $\mathfrak{A}$ is a free unital Banach $\mathfrak{A}$-module. The Banach direct sum of n copies of the module $\mathfrak{A}$ is also free unital Banach $\mathfrak{A}$-module since one has the topological isomorphism of unital Banach $\mathfrak{A}$-modules:

$$\bigoplus_1 \mathfrak{A} \cong \mathfrak{A}\hat{\otimes}\mathbb{C}^n. \tag{1}$$

Here the symbol $\bigoplus\limits_1$ denotes the l_1-sum (see, for example, [21]).

2. FREE $\mathfrak{A}_e$-MODULE

In what follows, G is an arbitrary group. As in the monoid S, the identity element of G is denoted by the letter e.

We suppose that there exists a surjective homomorphism of monoids

$$\sigma : S \longrightarrow G.$$

To obtain the results of the note we need the topological grading of the semigroup C^*-algebra $C_r^*(S)$ over G which was constructed in [18]. The definitions of graded and topologically graded C^*-algebras are contained in [19, §§ 16.2, 19.2]. Further we briefly describe the construction which allows us to set a grading on the C^*-algebra $C_r^*(S)$.

For every element $a \in S$ we consider two symbols T_a^{-1} and T_a^1. We denote by F the free semigroup over the alphabet $\{T_a^{-1}, T_a^1 \mid a \in S\}$. The semigroup F is involutive. An element V of this semigroup is a word (*monomial*) of the form

$$V = T_{a_1}^{i_1} T_{a_2}^{i_2} ... T_{a_k}^{i_k}, \tag{2}$$

where $a_1, ..., a_k \in S,\ i_1, ..., i_k \in \{-1, 1\}, k \in \mathbb{N}$. The involution operation on the semigroup F is given by

$$V^* = T_{a_k}^{1-i_k} T_{a_{k-1}}^{1-i_{k-1}} ... T_{a_1}^{1-i_1}.$$

We define the mapping $\mathrm{ind} : F \longrightarrow G$ by the formula

$$\mathrm{ind}\,(V) = \sigma(a_1)^{i_1} \sigma(a_2)^{i_2} ... \sigma(a_k)^{i_k}.$$

It is easily seen that the mapping ind is involutive surjective homomorphism of semigroups. The value $\mathrm{ind}\,(V)$ is called *the σ-index of monomial V*.

Every monomial V defines the bounded linear operator $\widehat{V}$ on the Hilbert space $l^2(S)$ as follows:

$$\widehat{T}_a^1 = T_a,\ \widehat{T}_a^{-1} = T_a^*,$$

and if V is a monomial of form (2) then we put

$$\widehat{V} = \widehat{T}_{a_1}^{i_1} \widehat{T}_{a_2}^{i_2} ... \widehat{T}_{a_k}^{i_k}.$$

We call $\widehat{V}$ *an operator monomial*.

In [18], it is shown that if two monomials define the same linear operator then the σ-indexes of these monomials coincide. Therefore the value $\mathrm{ind}\,(V) \in G$ is also called *the σ-index of an operator monomial $\widehat{V}$*.

It is straightforward to check that the set of all monomials with the σ-index e is an involutive subsemigroup in the semigroup of monomials F.

In the sequel, the symbol $\mathfrak{A}_e$ stands for the C^*-subalgebra generated by the set of all operator monomials with the σ-index e in the C^*-algebra $C_r^*(S)$.

For every $g \in G$, we denote by the symbol $\mathfrak{A}_g$ the Banach space which is defined as the closure of the linear hull for the set of all operator monomials with the σ-index g in the C^*-algebra $C_r^*(S)$.

The family of subspaces $\{\mathfrak{A}_g \mid g \in G\}$ constitutes a topological G-grading for the reduced semigroup C^*-algebra $C_r^*(S)$ [18, Theorem 2]. In the case of a finite group G, the underlying linear space of the C^*-algebra $C_r^*(S)$ is represented as the finite direct sum of its subspaces [18, Theorem 4]:

$$C_r^*(S) = \bigoplus_{g \in G} \mathfrak{A}_g. \tag{3}$$

It follows from equality (3) that each element $A \in C_r^*(S)$ has a unique representation in the form of the finite sum

$$A = \sum_{g \in G} A_g, \tag{4}$$

where $A_g \in \mathfrak{A}_g$.

Moreover, it is proved in [18, Lemma 5] that the space $\mathfrak{A}_g$ is a cyclic Banach $\mathfrak{A}_e$-module for each $g \in G$. In order to get a generator of the Banach $\mathfrak{A}_e$-module $\mathfrak{A}_g$, one takes an arbitrary element x_g from the set $\sigma^{-1}(g)$. Then we have the equality

$$\mathfrak{A}_g = \mathfrak{A}_e \cdot T_{x_g}. \tag{5}$$

The following theorem provides the condition under which the cyclic Banach $\mathfrak{A}_e$-module $\mathfrak{A}_g$ is a free $\mathfrak{A}_e$-module.

Theorem 1. *Let S be a cancellative monoid. Let G be a group with the identity e and $\sigma : S \longrightarrow G$ be a surjective homomorphism of monoids. For $g \in G$, let $\mathfrak{A}_g$ be the closed linear hull for the set of all operator monomials with the σ-index g in the reduced semigroup C^*-algebra $C^*_r(S)$. If there exists an element $x_g \in \sigma^{-1}(g)$ which is invertible in the monoid S then the cyclic Banach $\mathfrak{A}_e$-module $\mathfrak{A}_g$ is topologically isomorphic to the Banach $\mathfrak{A}_e$-module $\mathfrak{A}_e$.*

Proof. Let $x_g \in \sigma^{-1}(g)$ be an invertible element in the monoid S. We define the morphism of Banach $\mathfrak{A}_e$-modules as follows:

$$\varphi : \mathfrak{A}_e \longrightarrow \mathfrak{A}_e \cdot T_{x_g} : A \mapsto A \cdot T_{x_g}.$$

Since equality (5) holds, the module $\mathfrak{A}_g$ is topologically isomorphic to the quotient module $\mathfrak{A}_e / \ker \varphi$ [20, Proposition VI.2.3].

We claim that $\ker \varphi = \{0\}$. Indeed, let us suppose that $A \cdot T_{x_g} = B \cdot T_{x_g}$ for $A, B \in \mathfrak{A}_e$. Denote by $x_g^{-1} \in S$ the inverse element of x_g. To obtain a contradiction, we assume that $A \neq B$. Then there exists an element $a \in S$ such that $Ae_a \neq Be_a$. But, on the other hand, one has the equality $A \cdot T_{x_g} e_{x_g^{-1}a} = B \cdot T_{x_g} e_{x_g^{-1}a}$, which implies $Ae_a = Be_a$. Thus we have the contradiction. Hence, $\ker \varphi = \{0\}$, as claimed.

Therefore there exists a topological isomorphism $\mathfrak{A}_g \cong \mathfrak{A}_e$ of Banach $\mathfrak{A}_e$-modules. □

Further, we consider a set $X \subset S$ such that for every $g \in G$ there exists a unique element $x \in X$ satisfying the condition $X \cap \sigma^{-1}(g) = \{x\}$. We call X *a set of representatives for the preimages* $\sigma^{-1}(g)$, where $g \in G$. In [18], it is proved that the C^*-algebra $C^*_r(S)$ is a Banach $\mathfrak{A}_e$-module with the generating set $\{T_x \mid x \in X\}$.

The following theorem contains sufficient conditions under which the $\mathfrak{A}_e$-module $C^*_r(S)$ is a free Banach $\mathfrak{A}_e$-module.

Theorem 2. *Let S be a cancellative monoid. Let G be a finite group with the identity e and $\sigma : S \longrightarrow G$ be a surjective homomorphism of monoids. Let $\mathfrak{A}_e$ be the C^*-subalgebra in the C^*-algebra $C^*_r(S)$ generated by all operator monomials with the σ-index e. If there exists a set X of representatives for the preimages $\sigma^{-1}(g)$, $g \in G$, which is contained in a subgroup of the monoid S, then the $\mathfrak{A}_e$-module $C^*_r(S)$ is a free Banach $\mathfrak{A}_e$-module.*

Proof. Under the hypotheses of the theorem, we shall show that there is a topological isomorphism

$$C^*_r(S) \cong \mathfrak{A}_e \hat{\otimes} \mathbb{C}^n$$

of Banach $\mathfrak{A}_e$-modules, where n is an order of the group G. To do this, by (1) and (3), it is sufficient to prove that there exists a topological isomorphism

$$\bigoplus_{g \in G} \mathfrak{A}_g \cong \bigoplus_{1} \mathfrak{A}_e \tag{6}$$

between the Banach $\mathfrak{A}_e$-modules. On the right-hand side of (6), the number of summands in the direct l_1-sum is equal to the order of the group G. Below we denote an arbitrary element of this sum by a tuple $B = (B_g)_{g \in G}$, whose norm is given by $||B||_1 = \sum\limits_{g \in G} ||B_g||$, where $B_g \in \mathfrak{A}_e$. On the left-hand side of (6), every element of the direct sum of linear subspaces is written as sum (4).

To construct isomorphism (6), we take a set X of representatives such that each $x \in X$ possesses the inverse element in the monoid S. Then, by Theorem 1, for every $g \in G$ there exists a topological isomorphism of Banach $\mathfrak{A}_e$-modules

$$\psi_g : \mathfrak{A}_e \longrightarrow \mathfrak{A}_g.$$

Using the topological isomorphisms ψ_g, we define the linear operator

$$\alpha : \bigoplus_1 \mathfrak{A}_e \longrightarrow \bigoplus_{g\in G} \mathfrak{A}_g$$

by the formula $\alpha(B) = \sum\limits_{g\in G} \psi_g(B_g)$.

It is straightforward to check that the operator α is surjective. The linear independence of the family of subspaces $\{\mathfrak{A}_g\}_{g\in G}$ implies the injectivity of the operator α.

The continuity of the operator α follows from the chain of the inequalities

$$||\alpha(B)|| \leq \sum_{g\in G} ||\psi_g(B_g)|| \leq \max_{g\in G} ||\psi_g|| \sum_{g\in G} ||B_g|| = \max_{g\in G} ||\psi_g|| \, ||B||_1.$$

By the Banach inverse operator theorem, since α is a bijective bounded linear operator between Banach spaces, its inverse linear operator

$$\alpha^{-1} : \bigoplus_{g\in G} \mathfrak{A}_g \longrightarrow \bigoplus_1 \mathfrak{A}_e$$

is bounded as well.

Obviously, both operators α and α^{-1} are morphisms of left $\mathfrak{A}_e$-modules. Thus the operator α is a topological isomorphism of $\mathfrak{A}_e$-modules.

Finally, we conclude that the C^*-algebra $C_r^*(S)$ is a free Banach $\mathfrak{A}_e$-module, as required. □

ACKNOWLEDGMENTS

The author thanks Professor S.A. Grigoryan for drawing her attention to the subject under consideration.

REFERENCES

1. L. A. Coburn, "The C^*-algebra generated by an isometry," Bull. Am. Math. Soc. **73**, 722–726 (1967).
2. L. A. Coburn, "The C^*-algebra generated by an isometry. II," Trans. Am. Math. Soc. **137**, 211–217 (1969).
3. R. G. Douglas, "On the C^*-algebra of a one-parameter semigroup of isometries," Acta Math. **128**, 143–152 (1972).
4. G. J. Murphy, "Ordered groups and Toeplitz algebras," J. Oper. Theory **18**, 303–326 (1987).
5. G. J. Murphy, "Toeplitz operators and algebras," Math. Z. **208**, 355–362 (1991).
6. X. Li, "Semigroup C^*-algebras," in *Operator Algebras and Applications, The Abel Symposium 2015,* Ed. by T. M. Carlsen, N. S. Larsen, S. Neshveyev, and Ch. Skau (Springer, Cham, 2016), p. 191.
7. M. A. Aukhadiev, S. A. Grigoryan, and E. V. Lipacheva, "A compact quantum semigroup generated by an isometry," Russ. Math. (Iz. VUZ) **55**, 78 (2011).
8. E. V. Lipacheva and K. H. Hovsepyan, "The structure of C*-subalgebras of the Toeplitz algebra fixed with respect to a finite group of automorphisms," Russ. Math. (Iz. VUZ) **59** (6), 10–17 (2015).
9. E. V. Lipacheva and K. H. Hovsepyan, "The structure of invariant ideals of some subalebras of Toeplitz algebra," J. Contemp. Math. Anal. **50** (2), 70–79 (2015).
10. S. A. Grigorian and E. V. Lipacheva, "On the structure of C^*-algebras generated by representations of an elementary inverse semigroup," Uch. Zap. Kazan. Univ., Ser. Fiz.-Mat. Nauki **158**, 180–193 (2016).
11. S. A. Grigoryan, T. A. Grigoryan, E. V. Lipacheva, and A. S. Sitdikov, "C^*-algebra generated by the path semigroup," Lobachevskii J. Math. **37**, 740–748 (2016).
12. E. V. Lipacheva, "On a class of graded ideals of semigroup C*-algebras," Russ. Math. (Iz. VUZ) **62** (10), 37–46 (2018).
13. E. V. Lipacheva, "Embedding semigroup C*-algebras into inductive limits," Lobachevskii J. Math. **40**, 667–675 (2019).
14. S. A. Grigoryan, E. V. Lipacheva, and A. S. Sitdikov, "Nets of graded C^*-algebras over partially ordered sets," SPb. Math. J. **30**, 901–915 (2019).

15. R. N. Gumerov, E. V. Lipacheva, and T. A. Grigoryan, "On a topology and limits for inductive systems of C^*-algebras," Int. J. Theor. Phys. **60**, 499–511 (2021).
16. S. A. Grigoryan, R. N. Gumerov, and E. V. Lipacheva, "On extensions of semigroups and their applications to Toeplitz algebras," Lobachevskii J. Math. **40**, 2052–2061 (2019).
17. R. N. Gumerov and E. V. Lipacheva, "Topological grading of semigroup C*-algebras," Herald of the Bauman Moscow State Technical University, Series Natural Sciences **90** (3), 44–55 (2020).
18. E. V. Lipacheva, "On graded semigroup C^*-algebras and Hilbert modules," Proc. Steklov Inst. Math. **313**, 120–130 (2021).
19. R. Exel, *Partial Dynamical Systems, Fell Bundles and Applications,* vol. 224 of *Math. Surv. Monograph* (Am. Math. Soc., Providence, RI, 2017).
20. A. Ya. Helemskii, *Banach and Locally Convex Algebras* (Oxford Science, Clarendon, New York, 1993).
21. A. Ya. Helemskii, *Lectures and Exercises in Functional Analysis* (AMS, Providence, RI, 2006).