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On Graded Semigroup C^*-Algebras and Hilbert Modules

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Abstract—Reduced semigroup C^*-algebras for arbitrary cancellative semigroups are studied. It is proved that if there exists a semigroup epimorphism from a semigroup to an arbitrary group G, then the corresponding semigroup C^*-algebra is topologically G-graded. It is also demonstrated that if the group is finite, then the graded semigroup C^*-algebra has the structure of a projective Hilbert C^*-module.

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INTRODUCTION

In this paper, we consider reduced semigroup C^*-algebras, namely, the algebras generated by left regular representations of cancellative semigroups. The study of such C^*-algebras was initiated by Coburn [2, 3], Douglas [4], and Murphy [21, 22]. The further development of the theory of semigroup C^*-algebras is due to many authors (see, for example, the references cited in [16]).

The present work is a continuation of the studies of reduced semigroup C^*-algebras started in [1, 8–13, 17–19]. Here we study the following problems: the construction of a grading for reduced semigroup C^*-algebras, and the existence of a Hilbert C^*-module structure on a graded semigroup C^*-algebra.

The grading of an object of a category makes it possible to better understand the structure of this object. In the category of C^*-algebras, one uses C^*-bundles, or Fell bundles, to construct a grading. These bundles were introduced by Fell [7], who employed them to extend the notions of harmonic analysis to the noncommutative case. Exel [5] introduced the notion of a topologically graded C^*-algebra. A refinement of the definition of such an algebra is contained in [24]. It is important that the topological grading of a C^*-algebra guarantees the existence of special operators which are analogs of the Fourier coefficients. For a detailed account of the theory of graded C^*-algebras, we refer the reader to the monograph [6]. In [1, 8, 9, 11, 17], the authors studied questions related to the construction of gradings for different semigroup C^*-algebras. In [11], the notion of the σ-index of a monomial was introduced and used to construct a G-grading of a semigroup C^*-algebra for a finite cyclic group G. In the present study, we employ this notion to construct a G-grading of a semigroup C^*-algebra for an arbitrary group G, including a nonabelian one.

It is well known that the theory of Hilbert C^*-modules is a convenient tool for studying C^*-algebras. A detailed introduction to the theory of Hilbert C^*-modules, as well as many of its applications, is contained in the book [20]. In [23], the authors used the theory of Hilbert C^*-modules to define a noncommutative analog of branched coverings. To define the structure of a Hilbert C^*-module on a semigroup C^*-algebra, we use the conditional expectation of algebraically finite index (see [25]) and thus define a noncommutative covering of a semigroup C^*-algebra [23, Definition 1.4].

The paper consists of the introduction and four sections. In Section 1, we give necessary information on semigroup C^*-algebras and the definitions of graded and topologically graded C^*-algebras. We present all necessary definitions related to Banach and Hilbert modules over C^*-algebras.

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In Sections 2 and 3, we construct a grading and show that this grading is topological. The main result of these two sections is formulated as follows (Theorem 2). If there exists a semigroup epimorphism $\sigma\colon S \to G$ from an arbitrary cancellative semigroup S to an arbitrary group G, then the semigroup C^*-algebra $C^*_{\mathrm{r}}(S)$ is a topologically G-graded C^*-algebra. This theorem generalizes the result obtained in [11], where a similar statement was proved for an abelian semigroup and a finite cyclic group.

In Section 4, we study the existence of a Hilbert C^*-module structure on the G-graded C^*-algebra $C^*_{\mathrm{r}}(S)$ under consideration. Namely, we prove that if G is a finite group, then the C^*-algebra $C^*_{\mathrm{r}}(S)$ has the structure of a projective Hilbert C^*-module.

1. PRELIMINARIES

Throughout the paper, we denote by S an arbitrary cancellative semigroup with identity. Denote the identity in S by e.

The object of the present study is the reduced semigroup C^*-algebra $C^*_{\mathrm{r}}(S)$. In this connection, we recall the definition of this C^*-algebra.

Consider the Hilbert space $l^2(S)$ of square summable complex-valued functions defined on S:

$$l^2(S) := \left\{ f\colon\ S \to \mathbb{C} \;\middle|\; \sum_{a\in S} |f(a)|^2 < +\infty \right\}.$$

Denote the canonical orthonormal basis of $l^2(S)$ by $\{e_a \mid a \in S\}$, where

$$e_a(b) := \begin{cases} 1 & \text{if } \ a = b, \\ 0 & \text{if } \ a \neq b. \end{cases}$$

In the algebra of all bounded operators on $l^2(S)$, consider the C^*-subalgebra generated by the set of isometries $\{T_a \mid a \in S\}$, where the operator T_a is defined by the formula

$$T_a(e_b) = e_{ab}, \qquad a, b \in S.$$

It is this C^*-subalgebra that is called the reduced semigroup C^*-algebra $C^*_{\mathrm{r}}(S)$.

Next, for the convenience of the reader, we recall the definition of a G-graded C^*-algebra from [6, Definition 16.2].

Let $\mathfrak{A}$ be an arbitrary C^*-algebra and G be a group. Then $\mathfrak{A}$ is said to be a *G-graded* C^*-algebra if there exists a family of linearly independent closed subspaces $\{\mathfrak{A}_g \subset \mathfrak{A}\}_{g\in G}$ such that the following properties hold for any $g, h \in G$:

(1) $\mathfrak{A}_g\mathfrak{A}_h \subset \mathfrak{A}_{gh}$,

(2) $\mathfrak{A}_g^* = \mathfrak{A}_{g^{-1}}$, and

(3) $\mathfrak{A} = \overline{\bigoplus_{g\in G} \mathfrak{A}_g}$.

In this case, the family of Banach spaces $\{\mathfrak{A}_g\}_{g\in G}$ is called a *C^*-algebraic bundle* or a *Fell bundle* over the group G.

In [6, Definition 19.2], Exel also considered the notion of grading in a stronger sense. Namely, a G-graded C^*-algebra $\mathfrak{A}$ is said to be *topologically graded* if there exists a contractive linear map

$$F\colon\ \mathfrak{A} \to \mathfrak{A}_e$$

that coincides with the identity map on the subspace $\mathfrak{A}_e$, where e is the identity element of the group, and vanishes on every subspace $\mathfrak{A}_g$, where $g \in G$, $g \neq e$.

An important property of a topologically graded C^*-algebra is the existence of Fourier coefficients (see [6, Corollary 19.6]). This means that for every $g \in G$ there exists a contractive linear map

$$F_g\colon \mathfrak{A} \to \mathfrak{A}_g$$

such that the equality $F_g(A) = A_g$ holds for any finite sum $A = \sum_{h\in G} A_h$ with $A_h \in \mathfrak{A}_h$. Moreover, the maps F_g, $g \in G$, have the following property: the equalities

$$F_g(BA) = BF_{h^{-1}g}(A) \qquad \text{and} \qquad F_g(AB) = F_{gh^{-1}}(A)B$$

hold for any $B \in \mathfrak{A}_h$, $h \in G$, and $A \in \mathfrak{A}$.

The notions related to Banach and C^*-Hilbert modules are contained in the books [14, 20]. Recall the necessary definitions. By a module we mean a left module.

A module $\mathfrak{M}$ over a C^*-algebra $\mathfrak{A}$ is called a *Banach $\mathfrak{A}$-module* if it is a Banach space with norm satisfying the inequality $\|A \cdot M\| \le \|A\| \cdot \|M\|$, where $A \in \mathfrak{A}$ and $M \in \mathfrak{M}$. A subset X in a Banach $\mathfrak{A}$-module $\mathfrak{M}$ is called a *set of generators* if the finite $\mathfrak{A}$-linear combinations of elements of X are dense in $\mathfrak{M}$. If X is a finite set, then the module $\mathfrak{M}$ is said to be *finitely generated*.

An element M in an $\mathfrak{A}$-module $\mathfrak{M}$ is said to be *cyclic* if the following equality holds:

$$\mathfrak{M} = \mathfrak{A} \cdot M := \{A \cdot M \mid A \in \mathfrak{A}\}.$$

A module with a cyclic element is called a *cyclic module*.

A module $\mathfrak{M}$ over a C^*-algebra $\mathfrak{A}$ is called a *pre-Hilbert $\mathfrak{A}$-module* if it is equipped with a sesquilinear form $\langle\cdot,\cdot\rangle\colon \mathfrak{M} \times \mathfrak{M} \to \mathfrak{A}$, called an $\mathfrak{A}$-valued scalar (or inner) product, that has the following properties for any $M, N \in \mathfrak{M}$ and $A \in \mathfrak{A}$:

(1) $\langle M, M\rangle \ge 0$,

(2) $\langle M, M\rangle = 0$ if and only if $M = 0$,

(3) $\langle M, N\rangle = \langle N, M\rangle^*$, and

(4) $\langle A \cdot M, N\rangle = A\langle M, N\rangle$.

If $\mathfrak{M}$ is a pre-Hilbert $\mathfrak{A}$-module, then one can define a norm $\|\cdot\|_{\mathfrak{M}}$ on it as $\|M\|_{\mathfrak{M}} = \|\langle M, M\rangle\|^{1/2}$ for any $M \in \mathfrak{M}$ (see [20, Proposition 1.2.4]).

A pre-Hilbert $\mathfrak{A}$-module $\mathfrak{M}$ that is complete in the norm $\|\cdot\|_{\mathfrak{M}}$ is called a *Hilbert C^*-module*.

For a C^*-algebra $\mathfrak{A}$ and a C^*-subalgebra $\mathfrak{B}$ in $\mathfrak{A}$, a linear map $E\colon \mathfrak{A} \to \mathfrak{B}$ such that $E(B) = B$ for any $B \in \mathfrak{B}$ and $\|E(A)\| \le \|A\|$ for any $A \in \mathfrak{A}$ is called a *conditional expectation* (see [20, 25]). A conditional expectation is said to be *faithful* if for any positive $A \in \mathfrak{A}$ the equality $E(A) = 0$ implies $A = 0$. It is well known (see [20, Example 1.3.6]) that in this case one can introduce the structure of a left pre-Hilbert $\mathfrak{B}$-module on the C^*-algebra $\mathfrak{A}$ by defining the inner product

$$\langle A, B\rangle = E(AB^*) \qquad \text{for any} \quad A, B \in \mathfrak{A}.$$

2. GRADING OF THE C^*-ALGEBRA $C^*_{\mathrm{r}}(S)$

Let G be an arbitrary group. Denote the identity element of the group by e.

Suppose that there exists a surjective semigroup homomorphism

$$\sigma\colon S \to G. \tag{2.1}$$

Then the semigroup S can be represented as a disjoint union of subsets S_g,

$$S = \bigsqcup_{g\in G} S_g, \tag{2.2}$$

such that every S_g is the complete preimage of the element $g \in G$, i.e., $\sigma^{-1}(g) = S_g$.

To construct a grading of the semigroup C^*-algebra $C^*_{\mathrm{r}}(S)$, we introduce the notion of a monomial as well as the notion of a σ-index for a monomial and for the corresponding operator in this C^*-algebra, where σ is the surjective homomorphism (2.1). We use the construction that was first introduced by the author in the joint paper [11].

Consider the free semigroup generated by the set $\{T_a^{-1}, T_a^1 \mid a \in S\}$. The elements of this semigroup are words of the form

$$V = T_{a_k}^{i_k} T_{a_{k-1}}^{i_{k-1}} \dots T_{a_1}^{i_1}, \tag{2.3}$$

where $a_1, \dots, a_k \in S$ and $i_1, \dots, i_k \in \{-1, 1\}$. We will call these words *monomials*. The number k in (2.3) is called the *length* of the monomial. The semigroup itself is called the *monomial semigroup* and is denoted by Mon.

The monomial semigroup is an involutive semigroup. The involution is defined on a monomial of the form (2.3) by the formula

$$V^* = T_{a_1}^{-i_1} T_{a_2}^{-i_2} \dots T_{a_k}^{-i_k}.$$

Now we define a map of semigroups $\operatorname{ind}\colon \mathrm{Mon} \to G$. For a monomial of the form (2.3), we set by definition

$$\operatorname{ind} V = \sigma(a_k)^{i_k} \sigma(a_{k-1})^{i_{k-1}} \dots \sigma(a_1)^{i_1}.$$

It is easy to see that the equalities

$$\operatorname{ind}(V \cdot W) = \operatorname{ind} V \cdot \operatorname{ind} W \qquad \text{and} \qquad \operatorname{ind}(V^*) = (\operatorname{ind} V)^{-1} \tag{2.4}$$

hold for any $V, W \in \mathrm{Mon}$. Hence, the map ind is an involutive surjective semigroup homomorphism.

Every monomial V defines an operator $\widehat{V}$ on the Hilbert space $l^2(S)$ as follows:

$$\widehat{T}_a^1 = T_a, \qquad \widehat{T}_a^{-1} = T_a^*,$$

and if V is a monomial of the form (2.3), then

$$\widehat{V} = \widehat{T}_{a_k}^{i_k} \widehat{T}_{a_{k-1}}^{i_{k-1}} \dots \widehat{T}_{a_1}^{i_1}. \tag{2.5}$$

We will call operators of the form (2.5) *operator monomials*.

Lemma 1. *Let $V \in \mathrm{Mon}$ and $\widehat{V} e_a \neq 0$ for some basis vector $e_a \in l^2(S)$. Then there exists an element $b \in S$ such that the following equalities hold*:

$$\widehat{V} e_a = e_b \qquad \text{and} \qquad \sigma(b) = \operatorname{ind} V \cdot \sigma(a).$$

Proof. Let V be a monomial of the form (2.3). We will prove the lemma by induction on the length k of the monomial.

Let $k = 1$. Then two cases are possible: either $V = T_{a_1}^1$ or $V = T_{a_1}^{-1}$. Let $\widehat{V} e_a \neq 0$. Then, in the first case, we obtain $T_{a_1} e_a = e_{a_1 a}$. Hence, $b = a_1 a$ and

$$\sigma(b) = \sigma(a_1)\sigma(a) = \operatorname{ind} V \cdot \sigma(a).$$

In the second case, we have $T_{a_1}^* e_a = e_b$, where $a = a_1 b$. Therefore, $\sigma(a) = \sigma(a_1)\sigma(b)$, and so

$$\sigma(b) = (\sigma(a_1))^{-1} \sigma(a) = \operatorname{ind} V \cdot \sigma(a).$$

Now, consider a monomial V of arbitrary length k. Obviously, we can write it as $V = T_{a_k}^{i_k} V'$, where V' satisfies the equalities $\widehat{V}' e_a = e_{b'}$ and $\sigma(b') = \operatorname{ind} V' \cdot \sigma(a)$ by the induction hypothesis. Then we have $\widehat{V} e_a = \widehat{T}_{a_k}^{i_k} e_{b'} = e_b$. In the same way as in the case of $k = 1$, we obtain the equality $\sigma(b) = \operatorname{ind} T_{a_k}^{i_k} \cdot \sigma(b')$. This implies the required assertion

$$\sigma(b) = \operatorname{ind} T_{a_k}^{i_k} \cdot \operatorname{ind} V' \cdot \sigma(a) = \operatorname{ind} V \cdot \sigma(a). \qquad \square$$

It follows from Lemma 1 that if $\widehat{V}_1 = \widehat{V}_2$, then $\operatorname{ind} V_1 = \operatorname{ind} V_2$.

For an arbitrary monomial $V \in \mathrm{Mon}$, we refer to the value $\operatorname{ind} V$ of the map ind on V both as the *σ-index* of the monomial V and the *σ-index* of the operator monomial $\widehat{V}$.

Finite linear combinations of operator monomials form an involutive subalgebra, which is dense in the C^*-algebra $C^*_{\mathrm{r}}(S)$. Denote this subalgebra by $P(S)$.

It is easy to see that the monomials of σ-index e form an involutive subsemigroup in the monomial semigroup Mon. Denote by $\mathfrak{A}_e$ the C^*-subalgebra in $C^*_{\mathrm{r}}(S)$ generated by the operator monomials of σ-index e.

For any $g \in G$, the closure of the linear hull of the set of all operator monomials of σ-index g is a Banach subspace in the C^*-algebra $C^*_{\mathrm{r}}(S)$, which we denote by $\mathfrak{A}_g$.

Next, using the expansion (2.2), we represent the Hilbert space $l^2(S)$ as an orthogonal sum of subspaces:

$$l^2(S) = \bigoplus_{g \in G} H_g, \tag{2.6}$$

where every subspace H_g, $g \in G$, has a Hilbert basis given by the family of functions $\{e_a \mid a \in S_g\}$.

The following lemma shows how the spaces H_g behave under the action of the elements of the space $\mathfrak{A}_h$, for $g, h \in G$.

Lemma 2. *For any $g, h \in G$ and any operator $A \in \mathfrak{A}_h$, the following inclusion holds*:

$$A(H_g) \subset H_{hg}.$$

In particular, for any $g \in G$, the subspace H_g is invariant under the action of any element of the C^*-algebra $\mathfrak{A}_e$.

Proof. Since finite linear combinations of operator monomials of σ-index h form a dense subspace in the Banach space $\mathfrak{A}_h$, it suffices to prove the lemma for such operator monomials.

Fix elements $h, g \in G$. Let V be an arbitrary monomial of σ-index h. Take any basis vector $e_a \in H_g$ with $\widehat{V} e_a \neq 0$. If there is no such vector, then $\widehat{V}(H_g) = \{0\} \subset H_{hg}$. By Lemma 1, we obtain $\widehat{V} e_a = e_b$, where $b \in S$ is an element such that $\sigma(b) = \operatorname{ind} V \cdot \sigma(a)$. Since $\sigma(a) = g$, we have $\sigma(b) = hg$. Thus, $b \in S_{hg}$ and $e_b \in H_{hg}$. □

Next, we apply Lemma 2 in order to prove that the family of subspaces $\{\mathfrak{A}_g \mid g \in G\}$ is a C^*-algebraic bundle over the group G. This statement is a generalization of Lemma 3 from [11].

Lemma 3. *The following assertions hold for the system of subspaces $\{\mathfrak{A}_g \mid g \in G\}$*:

(1) $\mathfrak{A}_g \mathfrak{A}_h \subset \mathfrak{A}_{gh}$,

(2) $\mathfrak{A}_g^* = \mathfrak{A}_{g^{-1}}$,

(3) *the family $\{\mathfrak{A}_g \mid g \in G\}$ is a linearly independent system of closed subspaces in the C^*-algebra $C^*_{\mathrm{r}}(S)$, and*

(4) $C^*_{\mathrm{r}}(S) = \overline{\bigoplus_{g \in G} \mathfrak{A}_g}$.

Proof. For operator monomials, assertions (1) and (2) follow from equalities (2.4). In the general case, the assertions hold since the finite linear combinations of operator monomials of σ-index g are dense in the Banach space $\mathfrak{A}_g$.

Let us prove assertion (3). Let $A = \sum_{g \in G} A_g = 0$, where $A_g \in \mathfrak{A}_g$. Note that this sum contains a finite number of nonzero terms. We will show that then $A_g = 0$ for any g. Let $g_0 \in G$ be an element such that $A_{g_0} \neq 0$. Then we have the representation

$$A_{g_0} = -\sum_{g \in G,\, g \neq g_0} A_g.$$

By Lemma 2, the inclusion $A_{g_0}(H_h) \subset H_{g_0h}$ holds for any $h \in G$. This implies the relation

$$-\sum_{g\in G,\, g\neq g_0} A_g(H_h) \subset H_{g_0h}.$$

On the other hand, by the same Lemma 2, we have the inclusion

$$-\sum_{g\in G,\, g\neq g_0} A_g(H_h) \subset \bigoplus_{g\in G,\, g\neq g_0} H_{gh}.$$

Since $g_0h \neq gh$ and the subspaces H_g, $g \in G$, are orthogonal, we obtain the equality

$$H_{g_0h} \cap \bigoplus_{g\in G,\, g\neq g_0} H_{gh} = \{0\}.$$

This implies the relation

$$A_{g_0} = -\sum_{g\in G,\, g\neq g_0} A_g = 0.$$

Let us prove assertion (4). Note that the finite linear combinations of operator monomials are dense in the C^*-algebra $C^*_{\mathrm{r}}(S)$. On the other hand, they are dense in $\bigoplus_{g\in G} \mathfrak{A}_g$, since every such finite linear combination can be represented as $\sum_{g\in G} A_g$, where A_g is a finite linear combination of operator monomials of σ-index g. From the inclusion $\bigoplus_{g\in G} \mathfrak{A}_g \subset C^*_{\mathrm{r}}(S)$ we find that the subspace $\bigoplus_{g\in G} \mathfrak{A}_g$ is dense in $C^*_{\mathrm{r}}(S)$. $\square$

Thus, Lemma 3 implies the following result on the G-grading of the semigroup C^*-algebra $C^*_{\mathrm{r}}(S)$.

Theorem 1. *Let $\sigma\colon S \to G$ be a surjective semigroup homomorphism. Let $\mathfrak{A}_g$ be a closed subspace in $C^*_{\mathrm{r}}(S)$ generated by the operator monomials of σ-index g, where $g \in G$. Then the family of subspaces $\{\mathfrak{A}_g \mid g \in G\}$ forms a Fell bundle of the semigroup C^*-algebra $C^*_{\mathrm{r}}(S)$ over the group G.*

3. TOPOLOGICAL GRADING OF THE C^*-ALGEBRA $C^*_{\mathrm{r}}(S)$

Here we prove that the semigroup C^*-algebra $C^*_{\mathrm{r}}(S)$ is topologically graded (for the grading constructed in Section 2).

In the next lemma, we construct a linear bounded operator that allows us to talk about the topological G-grading of the semigroup C^*-algebra $C^*_{\mathrm{r}}(S)$.

Lemma 4. *There exists a contractive linear operator*

$$F\colon\ C^*_{\mathrm{r}}(S) \to \mathfrak{A}_e$$

that coincides with the identity operator on $\mathfrak{A}_e$ and vanishes on every subspace $\mathfrak{A}_g$, $g \in G$, $g \neq e$.

Proof. Recall that the involutive subalgebra $P(S)$, which consists of all finite linear combinations of operator monomials, is dense in the C^*-algebra $C^*_{\mathrm{r}}(S)$. Therefore, to prove the lemma, it suffices to construct a linear bounded operator

$$F\colon\ P(S) \to \mathfrak{A}_e$$

that leaves invariant the linear combinations of operator monomials of σ-index e and vanishes on the linear combinations of operator monomials of σ-index g for every $g \neq e$.

Since every element $A \in P(S)$ can be uniquely represented as a finite sum of nonzero terms of the form

$$A = \sum_{g\in G} A_g, \tag{3.1}$$

where $A_g \in \mathfrak{A}_g$, the formula

$$F(A) = A_e$$

obviously defines a linear operator on the normed space $P(S)$ that satisfies the required conditions.

Let us prove that F is a contractive operator. To this end, we fix an arbitrary element $A \in P(S)$ and consider its representation (3.1).

Let us show that the following estimate for the norms holds:

$$\|F(A)\| = \|A_e\| \leq \|A\|. \tag{3.2}$$

Recall that the Hilbert space $l^2(S)$ decomposes into the orthogonal sum (2.6). By Lemma 2, the subspaces H_g are invariant with respect to $\mathfrak{A}_e$. Therefore, for every element $A_e \in \mathfrak{A}_e$, we have the decomposition

$$A_e = \bigoplus_{g \in G} A_e^g,$$

where A_e^g denotes the restriction of the operator A_e to the subspace H_g. This implies the following equality for the operator norms:

$$\|A_e\| = \sup_{g \in G} \|A_e^g\|.$$

Fix an arbitrary number $\varepsilon > 0$. Let $g_0 \in G$ be an element such that the inequality

$$\|A_e^{g_0}\| \geq \|A_e\| - \varepsilon \tag{3.3}$$

is satisfied. Note that the following inequality holds:

$$\|A\| = \sup_{\|x\|=1,\ x \in l^2(S)} (Ax, Ax)^{1/2} \geq \sup_{\|x\|=1,\ x \in H_{g_0}} (Ax, Ax)^{1/2}.$$

For $x \in H_{g_0}$, consider the inner product

$$(Ax, Ax) = \left(\sum_{g \in G} A_g x, \sum_{h \in G} A_h x \right) = \sum_{g,h \in G} (A_g x, A_h x).$$

Let us show that $(A_g x, A_h x) = 0$ for $g \neq h$. Indeed, since $x \in H_{g_0}$, by Lemma 2 we have $A_g x \in H_{gg_0}$ and $A_h x \in H_{hg_0}$. If $g \neq h$, then $gg_0 \neq hg_0$, and since the subspaces H_{gg_0} and H_{hg_0} are orthogonal, we obtain $(A_g x, A_h x) = 0$. Thus, we have the estimate

$$(Ax, Ax) = \sum_{g \in G} (A_g x, A_g x) \geq (A_e x, A_e x),$$

which implies the inequality

$$\|A\| \geq \sup_{\|x\|=1,\ x \in H_{g_0}} (Ax, Ax)^{1/2} \geq \sup_{\|x\|=1,\ x \in H_{g_0}} (A_e x, A_e x)^{1/2} = \|A_e^{g_0}\|. \tag{3.4}$$

Since ε is arbitrary, from inequalities (3.3) and (3.4) we obtain the required inequality (3.2). $\square$

Note that the constructed linear operator

$$F\colon\ C_{\mathrm{r}}^*(S) \to \mathfrak{A}_e$$

is a conditional expectation. We will call it the *conditional expectation associated with the grading* $\{\mathfrak{A}_g \mid g \in G\}$.

Lemmas 3 and 4 allow us to claim that the C^*-algebra $C_{\mathrm{r}}^*(S)$ is topologically graded.

Theorem 2. *Let $\sigma\colon S \to G$ be a surjective semigroup homomorphism. Let $\mathfrak{A}_g$ be a closed subspace in $C_{\mathrm{r}}^*(S)$ generated by the operator monomials of σ-index g, where $g \in G$. Then the system of subspaces $\{\mathfrak{A}_g \mid g \in G\}$ forms a topological G-grading of the semigroup C^*-algebra $C_{\mathrm{r}}^*(S)$.*

4. FINITELY GENERATED PROJECTIVE HILBERT $\mathfrak{A}_e$-MODULE

In this section, we show that the C^*-algebra $C^*_{\mathrm{r}}(S)$ is a left Banach $\mathfrak{A}_e$-module. Moreover, if G is a finite group, then $C^*_{\mathrm{r}}(S)$ is a finitely generated projective Hilbert $\mathfrak{A}_e$-module.

Recall that the semigroup S can be represented as the disjoint union (2.2). Let us fix an arbitrary element x_g in each set S_g. Denote the set of all such x_g by X. Thus, $X \subset S$ and $X \cap S_g = \{x_g\}$ for any $g \in G$.

By a *set of representatives of the classes* $\{S_g \mid g \in G\} = \{\sigma^{-1}(g) \mid g \in G\}$ we will mean an arbitrary subset $X \subset S$ with the following property: for any $g \in G$, there exists a unique $x \in X$ such that $X \cap S_g = \{x\}$.

Lemma 5. *For every $g \in G$, the equality*

$$\mathfrak{A}_g = \mathfrak{A}_e \cdot T_{x_g}$$

holds; *i.e., the space $\mathfrak{A}_g$ is a cyclic Banach $\mathfrak{A}_e$-module, and the element T_{x_g} is a cyclic element of the module $\mathfrak{A}_g$.*

Proof. First, let us show that the inclusion $\mathfrak{A}_e \cdot T_{x_g} \subset \mathfrak{A}_g$ holds. If $A^{(e)} = \sum_i \alpha_i \widehat{V}_i^{(e)}$ is a finite linear combination of operator monomials of σ-index e, where $\alpha_i \in \mathbb{C}$, then $A^{(e)} T_{x_g} = \sum_i \alpha_i \widehat{V}_i^{(e)} T_{x_g}$ is obviously a finite linear combination of operator monomials of σ-index g. Let $\{A_n^{(e)}\}$ be a sequence of such linear combinations, and let $\lim_{n\to\infty} A_n^{(e)} = B_e \in \mathfrak{A}_e$. Then, in view of the inequality

$$\|A_n^{(e)} T_{x_g} - B_e T_{x_g}\| \le \|A_n^{(e)} - B_e\| \cdot \|T_{x_g}\| = \|A_n^{(e)} - B_e\|,$$

we obtain $B_e T_{x_g} = \lim_{n\to\infty} A_n^{(e)} T_{x_g} \in \mathfrak{A}_g$.

Let us establish the reverse inclusion $\mathfrak{A}_g \subset \mathfrak{A}_e \cdot T_{x_g}$. Let $B_g \in \mathfrak{A}_g$. Set $B_e := B_g T^*_{x_g}$. Then we obviously have $B_g = B_e T_{x_g}$. Just as above, taking into account that the σ-index of $T^*_{x_g}$ is equal to g^{-1}, we can show that $B_e \in \mathfrak{A}_e$. This completes the proof of the lemma. $\square$

Assertion (4) of Lemma 3 and Lemma 5 imply the following theorem.

Theorem 3. *Let G be an arbitrary group with identity e, $\sigma\colon S \to G$ be a surjective semigroup homomorphism, and X be a set of representatives of the classes $\{\sigma^{-1}(g) \mid g \in G\}$. Let $\mathfrak{A}_e$ be the C^*-subalgebra in $C^*_{\mathrm{r}}(S)$ generated by the operator monomials of σ-index e. Then the C^*-algebra $C^*_{\mathrm{r}}(S)$ is a Banach $\mathfrak{A}_e$-module with generating set $\{T_x \mid x \in X\}$.*

As pointed out in Section 1, since the system of subspaces $\{\mathfrak{A}_g \mid g \in G\}$ forms a topological G-grading of the C^*-algebra $C^*_{\mathrm{r}}(S)$, for every g there exists a contractive linear map

$$F_g\colon\ C^*_{\mathrm{r}}(S) \to \mathfrak{A}_g$$

such that for any finite sum $A = \sum_{g\in G} A_g$ with $A_g \in \mathfrak{A}_g$ we have the equality $F_g(A) = A_g$. Moreover, the maps F_g satisfy the equalities

$$F_g(AB) = F_{gh^{-1}}(A)B \qquad \text{and} \qquad F_g(BA) = BF_{h^{-1}g}(A) \tag{4.1}$$

for any $A \in C^*_{\mathrm{r}}(S)$ and $B \in \mathfrak{A}_h$.

Note that the construction of the operator F in Lemma 4 and the principle of extension by continuity (see, for example, [15, Ch. 2, § 1, Theorem 2]) immediately imply the equality $F_e = F$.

Next, let G be a finite group.

First, we examine the geometry of the underlying Banach space of the reduced semigroup C^*-algebra $C^*_{\mathrm{r}}(S)$.

Theorem 4. *Let G be a finite group with identity e, $\sigma\colon S \to G$ be a surjective semigroup homomorphism, and X be a set of representatives of the classes $\{\sigma^{-1}(g) \mid g \in G\}$. Let $\mathfrak{A}_e$ be the*

C^*-subalgebra in $C^*_{\mathrm{r}}(S)$ generated by the operator monomials of σ-index e. Then the C^*-algebra $C^*_{\mathrm{r}}(S)$ is a finitely generated Banach $\mathfrak{A}_e$-module with a set of generators $\{T_x \mid x \in X\}$. Moreover, it can be represented as a direct sum of a finite number of cyclic $\mathfrak{A}_e$-modules:

$$C^*_{\mathrm{r}}(S) = \bigoplus_{x\in X} \mathfrak{A}_e \cdot T_x.$$

Proof. First, we prove the equality of spaces

$$C^*_{\mathrm{r}}(S) = \bigoplus_{g\in G} \mathfrak{A}_g. \tag{4.2}$$

To this end, we show that the equality

$$A = \sum_{g\in G} F_g(A) \tag{4.3}$$

holds for any $A \in C^*_{\mathrm{r}}(S)$. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence in $P(S)$ that converges to A. Note that equality (4.3) holds for any finite sum of the form $A = \sum_{g\in G} A_g$; i.e., for any $A_n \in P(S)$, we have

$$A_n = \sum_{g\in G} (A_n)_g = \sum_{g\in G} F_g(A_n).$$

Then we obtain the chain of inequalities

$$\begin{aligned}\left\| A - \sum_{g\in G} F_g(A) \right\| &\le \|A - A_n\| + \left\| \sum_{g\in G} F_g(A_n) - \sum_{g\in G} F_g(A) \right\| \\ &\le \|A - A_n\| + \sum_{g\in G} \|F_g\| \cdot \|A_n - A\| = (k+1)\cdot \|A - A_n\|,\end{aligned}$$

where k is the order of the group G. This estimate implies equality (4.3).

Now, employing Lemma 5, we obtain the assertion of the theorem. □

In fact, as we will see below, if the group G is finite, then the C^*-algebra $C^*_{\mathrm{r}}(S)$ is a linearly generated projective C^*-Hilbert $\mathfrak{A}_e$-module.

Recall the definition of a finitely generated projective Hilbert module [20]. Let $\mathfrak{M}$ be a Hilbert $\mathfrak{A}$-module such that there exists a Hilbert $\mathfrak{A}$-module $\mathfrak{N}$ for which the direct sum $\mathfrak{M} \oplus \mathfrak{N}$ is isomorphic as a module to the direct sum of a finite number of copies of the Hilbert $\mathfrak{A}$-module $\mathfrak{A}$. Then $\mathfrak{M}$ is called a *finitely generated projective* $\mathfrak{A}$-module.

In [25, Corollary 3.1.4], conditions on the conditional expectation $E\colon \mathfrak{A} \to \mathfrak{B} \subset \mathfrak{A}$ are presented under which $\mathfrak{A}$ is a C^*-Hilbert module over the C^*-algebra $\mathfrak{B}$.

The conditional expectation $E\colon \mathfrak{A} \to \mathfrak{B} \subset \mathfrak{A}$ is called a conditional expectation of *algebraically finite index* (see [25, Definition 1.2.2] and [20, Sect. 4.5]) if there exists a finite set of elements $X_1, \dots, X_n \in \mathfrak{A}$ such that every element $A \in \mathfrak{A}$ can be represented as

$$A = \sum_{k=1}^{n} E(AX_k^*)X_k.$$

If the conditional expectation is a conditional expectation of algebraically finite index, then it is faithful [25]. In [25, Corollary 3.1.4], it is demonstrated that the algebraic finiteness of the index is equivalent to the fact that $\mathfrak{A}$ is a finitely generated projective Hilbert C^*-module over the C^*-algebra $\mathfrak{B}$ (see also [23, Theorem 5.7]).

In the following lemma and theorem, we prove that in the case of a finite group G the conditional expectation $F\colon C^*_{\mathrm{r}}(S) \to \mathfrak{A}_e$ constructed in Lemma 4 is a conditional expectation of algebraically finite index and that the C^*-algebra $C^*_{\mathrm{r}}(S)$ is a finitely generated projective Hilbert $\mathfrak{A}_e$-module.

Lemma 6. *The conditional expectation $F\colon C^*_{\mathrm{r}}(S) \to \mathfrak{A}_e$ associated with the grading $\{\mathfrak{A}_g \mid g \in G\}$ is a conditional expectation of algebraically finite index.*

Proof. By Theorem 4, any element $A \in C^*_{\mathrm{r}}(S)$ can be represented as the finite sum (4.3). Then, using the properties (4.1) and taking into account that $F_e = F$, we obtain for any $A \in C^*_{\mathrm{r}}(G)$ the equalities

$$A = \sum_{g\in G} F_g(A) = \sum_{g\in G} F_g(AT^*_{x_g}T_{x_g}) = \sum_{g\in G} F(AT^*_{x_g})T_{x_g}.$$

This means that the conditional expectation F constructed in Lemma 4 is a conditional expectation of algebraically finite index. $\square$

Theorem 5. *Let G be a finite group with identity e and $\sigma\colon S \to G$ be a surjective semigroup homomorphism. Let $\mathfrak{A}_e$ be the C^*-subalgebra in $C^*_{\mathrm{r}}(S)$ generated by the operator monomials of σ-index e. Then the C^*-algebra $C^*_{\mathrm{r}}(S)$ is a finitely generated projective Hilbert $\mathfrak{A}_e$-module.*

Proof. It follows from Lemma 6 that the conditional expectation $F\colon C^*_{\mathrm{r}}(S) \to \mathfrak{A}_e$ associated with the grading $\{\mathfrak{A}_g \mid g \in G\}$ is faithful. Then we can define the following $\mathfrak{A}_e$-valued inner product on the C^*-algebra $C^*_{\mathrm{r}}(S)$:

$$\langle A, B\rangle = F(AB^*),$$

where $A, B \in C^*_{\mathrm{r}}(S)$. On the other hand, as already mentioned above, the algebraic finiteness of the index of the conditional expectation F associated with the grading $\{\mathfrak{A}_g \mid g \in G\}$ implies that the C^*-algebra $C^*_{\mathrm{r}}(S)$ is a finitely generated projective Hilbert C^*-module over the C^*-algebra $\mathfrak{A}_e$. $\square$

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