ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2021, Vol. 313, pp. 60–69. © Pleiades Publishing, Ltd., 2021.
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Limits of Inductive Sequences of Toeplitz–Cuntz Algebras

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Received July 31, 2020; revised September 19, 2020; accepted December 12, 2020

Abstract—We consider inductive sequences of Toeplitz–Cuntz algebras. The connecting homomorphisms of such a sequence are defined by a finite set of sequences of positive integers. We prove that the inductive limit of such a sequence of Toeplitz–Cuntz algebras is isomorphic to the reduced semigroup C^*-algebra constructed for the unitalization of the free product of a finite family of semigroups of positive rational numbers. We show that the limit of the inductive sequence of Toeplitz–Cuntz algebras defined by a finite set of sequences of positive integers is a simple C^*-algebra.

DOI: 10.1134/S0081543821020073

INTRODUCTION

This paper is devoted to the study of inductive limits of sequences of Toeplitz–Cuntz algebras. Recall that a Toeplitz–Cuntz algebra $\mathcal{TO}_n$ is a unital C^*-algebra defined as a universal C^*-algebra generated by a set of n isometries with mutually orthogonal images. This algebra was defined and studied by Cuntz [4, 5]. Subsequently, the properties of Toeplitz–Cuntz algebras and their applications have been studied by many authors (see, for example, [1, 3, 6, 15, 16]). Note that these algebras turned out to be closely related to the normal involutive endomorphisms of the C^*-algebra of all bounded operators on a Hilbert space.

This paper continues the authors' studies on the theory of semigroup C^*-algebras and inductive systems of C^*-algebras, which were started in [2, 7–13, 17, 18]. In particular, the properties of inductive sequences of Toeplitz algebras defined by sequences of positive integers and of their limits were studied in the indicated papers.

In the present study, we consider inductive sequences consisting of copies of the same Toeplitz–Cuntz algebra $\mathcal{TO}_n$. The connecting homomorphisms of such a sequence are defined by means of a set of n sequences of positive integers $P_1,\dots,P_n$. For each sequence P_k, we construct a semigroup S_k isomorphic to a subsemigroup of the group of all rational numbers. We prove that the limit of the inductive sequence of Toeplitz–Cuntz algebras $\mathcal{TO}_n$ defined by the set $P_1,\dots,P_n$ is isomorphic to the reduced semigroup C^*-algebra $C^*_{\mathrm{r}}(S)$ constructed for the unitalization S of the free product of semigroups $S_1*\dots*S_n$. We also show that this inductive limit is a simple C^*-algebra.

The paper consists of the introduction and two sections. In Section 1, we present necessary notation, definitions, and preliminary information from the theories of C^*-algebras and semigroups. Section 2 contains results on the limits of inductive sequences of C^*-algebras.

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1. NOTATION AND PRELIMINARY INFORMATION

We begin with the definition of the Toeplitz–Cuntz algebra. Throughout the paper, $n \geq 2$ is a fixed positive integer. Let $U_1, \ldots, U_n$ be bounded linear operators on some Hilbert space that satisfy, together with the identity operator I on this space, the following conditions:

(1) $U_k^* U_k = I$ for any positive integer $k \leq n$;

(2) $U_k^* U_l = 0$ for any positive integers $k, l \leq n$ such that $k \neq l$;

(3) $\sum_{k=1}^{n} U_k U_k^* < I$.

Cuntz proved [5, Lemma 3.1] that the C^*-algebra generated by the operators $U_1, \ldots, U_n$ satisfying relations (1)–(3) is independent of the choice of these elements and, hence, can be characterized as a universal C^*-algebra generated by n operators $U_1, \ldots, U_n$ and relations (1)–(3). As mentioned in the Introduction, this universal C^*-algebra is called the *Toeplitz–Cuntz algebra* and denoted by $\mathcal{TO}_n$.

Recall how the free product of a disjoint family of semigroups is constructed. Let S_k, $1 \leq k \leq n$, be a set of semigroups such that $S_i \cap S_j = \varnothing$ for $i \neq j$. Denote by $S_1 * \ldots * S_n$ the set of all nonempty finite sequences $a_1 \ldots a_l$, $l \in \mathbb{N}$, consisting of elements of the disjoint union $\bigsqcup_{k=1}^{n} S_k$ such that, for any index $1 \leq i \leq l-1$, the condition $a_i \in S_k$ for some $1 \leq k \leq n$ implies that $a_{i+1} \notin S_k$. In other words, in a sequence $a_1 \ldots a_l \in S_1 * \ldots * S_n$, any two adjacent elements belong to different semigroups. Next, introduce a binary operation $*$ on the set $S_1 * \ldots * S_n$ by the formula

$$a_1 \ldots a_l * b_1 \ldots b_m = \begin{cases} a_1 \ldots a_l b_1 \ldots b_m & \text{if } \ a_l \in S_i, \ b_1 \in S_j, \ i \neq j, \\ a_1 \ldots a_{l-1}(a_l \cdot b_1) b_2 \ldots b_m & \text{if } \ a_l, b_1 \in S_i \ \text{ for some } i, \end{cases}$$

where $a_1 \ldots a_l, b_1 \ldots b_m \in S_1 * \ldots * S_n$. One can easily check that the set $S_1 * \ldots * S_n$ equipped with the operation $*$ is a semigroup, which is called the *free product of the semigroups* S_k, $1 \leq k \leq n$. Note also that in the category of semigroups and their morphisms, the free product $S_1 * \ldots * S_n$ is defined as the coproduct of the family of objects S_k, $1 \leq k \leq n$ (see, for example, [14, Ch. VII, Sect. 1] for details).

Let us proceed to the definition of the reduced semigroup C^*-algebra $C_{\mathrm{r}}^*(S)$, where S is an arbitrary semigroup with left cancellation and neutral element.

Below, as usual, we denote by $l^2(S)$ the Hilbert space of all square integrable complex-valued functions on the semigroup S. Consider a standard orthonormal basis $\{e_a \mid a \in S\}$ of the Hilbert space $l^2(S)$, where the function e_a is defined by

$$e_a(b) := \begin{cases} 1 & \text{if } \ a = b, \\ 0 & \text{if } \ a \neq b. \end{cases}$$

For every $a \in S$ in the C^*-algebra $B(l^2(S))$ of all bounded operators on the Hilbert space $l^2(S)$, define an isometric operator $V_a \colon l^2(S) \to l^2(S)$ by the formula

$$V_a(e_b) = e_{ab}, \qquad b \in S.$$

Now, consider two subalgebras in the C^*-algebra $B(l^2(S))$. By $P(S)$ denote the involutive subalgebra generated by the set of isometries $\{V_a \mid a \in S\}$, and by $C_{\mathrm{r}}^*(S)$ denote the C^*-subalgebra in $B(l^2(S))$ obtained by completing $P(S)$ in the operator norm; the subalgebra $C_{\mathrm{r}}^*(S)$ is called the *reduced semigroup C^*-algebra for the semigroup S*. Every element of $P(S)$ can be represented as a linear combination with complex coefficients of operators of the form

$$V_{a_1}^{i_1} V_{a_2}^{i_2} \ldots V_{a_t}^{i_t}, \tag{1.1}$$

where $a_j \in S$, $i_j \in \{0, 1\}$, $j = 1, \ldots, t$, $t \in \mathbb{N}$, and we set $V_{a_j}^0 := V_{a_j}^*$ and $V_{a_j}^1 := V_{a_j}$.

Next, recall the definitions of an inductive sequence and inductive limit in the category of C^*-algebras and their $*$-homomorphisms (see, for example, [19, Ch. 6] for details).

An *inductive,* or *direct, sequence* is a set $\{A_k, \varphi_k\}_{k=1}^{+\infty}$ consisting of C^*-algebras A_k and $*$-homomorphisms $\varphi_k\colon A_k \to A_{k+1}$. To represent this inductive sequence, one often uses the diagram

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \dots . \tag{1.2}$$

An *inductive,* or *direct, limit* of the inductive sequence (1.2) is a pair $(A, \{\varphi_{k,\infty}\}_{k=1}^{+\infty})$ consisting of a C^*-algebra A and a sequence of $*$-homomorphisms $\{\varphi_{k,\infty}\colon A_k \to A\}_{k=1}^{+\infty}$ with the following properties:

(i) for any $k \in \mathbb{N}$, the diagram

$$\begin{array}{ccc} A_k & \xrightarrow{\quad\varphi_k\quad} & A_{k+1} \\ & \searrow^{\varphi_{k,\infty}} \quad {}^{\varphi_{k+1,\infty}}\swarrow & \\ & A & \end{array}$$

is commutative, i.e., $\varphi_{k,\infty} = \varphi_{k+1,\infty} \circ \varphi_k$;

(ii) (*universality property*) for any C^*-algebra B and any sequence of $*$-homomorphisms $\{\psi_{k,\infty}\colon A_k \to B\}_{k=1}^{+\infty}$ satisfying the condition $\psi_{k,\infty} = \psi_{k+1,\infty} \circ \varphi_k$ for every $k \in \mathbb{N}$, there exists a unique morphism $\psi\colon A \to B$ such that the diagram

$$\begin{array}{ccc} & A_k & \\ {}^{\varphi_{k,\infty}}\swarrow & & \searrow^{\psi_{k,\infty}} \\ A & \xrightarrow{\quad\psi\quad} & B \end{array}$$

is commutative, i.e., $\psi_{k,\infty} = \psi \circ \varphi_{k,\infty}$, for any $k \in \mathbb{N}$.

Often, the object A itself is called the inductive limit. It is well known that inductive limits can always be constructed in the category of C^*-algebras and their $*$-homomorphisms. For an inductive sequence $\{A_k, \varphi_k\}_{k=1}^{+\infty}$, its inductive limit is denoted by $\varinjlim\{A_k, \varphi_k\}$. Moreover, for every inductive sequence in the category of C^*-algebras and their $*$-homomorphisms, there exists a unique, up to isomorphism, inductive limit of this sequence.

Below we will need the following example of an inductive sequence of C^*-algebras and their $*$-homomorphisms. Let $\{A_k\}_{k=1}^{+\infty}$ be an increasing sequence of C^*-subalgebras of some C^*-algebra C, i.e., $A_k \subset A_{k+1}$ for any $k \in \mathbb{N}$. Then the sequence $\{A_k, \varphi_k\}_{k=1}^{+\infty}$, where each connecting $*$-homomorphism $\varphi_k\colon A_k \to A_{k+1}$ is a natural embedding, is an inductive sequence of C^*-subalgebras. Up to isomorphism, the inductive limit of the sequence $\{A_k, \varphi_k\}_{k=1}^{+\infty}$ is the pair $(A, \{\varphi_{k,\infty}\}_{k=1}^{+\infty})$ with

$$A = \overline{\bigcup_{k=1}^{+\infty} A_k}$$

and the $*$-homomorphism $\varphi_{k,\infty}\colon A_k \to A$ given by the natural embedding of C^*-algebras for every $k \in \mathbb{N}$ (see [19, Remark 6.1.3]).

2. INDUCTIVE SEQUENCES AND THEIR LIMITS

In this section, we consider an inductive sequence of C^*-algebras consisting of copies of the Toeplitz–Cuntz algebra $\mathcal{TO}_n$ for a fixed number n. The connecting morphisms of this sequence are defined by the exponentiation of the generating isometries of the C^*-algebra $\mathcal{TO}_n$, with exponents given by n sequences of positive integers

$$P_1 = \{p_{11}, p_{21}, \dots\}, \quad \dots, \quad P_n = \{p_{1n}, p_{2n}, \dots\} \tag{2.1}$$

such that for any $k \in \mathbb{N}$ there exists an index i, $1 \le i \le n$, for which $p_{ki} \ne 1$.

The following lemma will allow us to construct inductive sequences of Toeplitz–Cuntz algebras.

Lemma 1. *For every $k \in \mathbb{N}$, there exists a unique injective $*$-homomorphism*

$$\varphi_k \colon \mathcal{TO}_n \to \mathcal{TO}_n$$

defined by its action on the generating isometries $U_1, \ldots, U_n$ as follows:

$$\varphi_k(U_1) = U_1^{p_{k1}}, \quad \ldots, \quad \varphi_k(U_n) = U_n^{p_{kn}}.$$

Proof. Since $\mathcal{TO}_n$ is a universal C^*-algebra on n generators satisfying relations (1)–(3) from the definition of the Toeplitz–Cuntz algebra $\mathcal{TO}_n$, we should verify that the elements $U_1^{p_{k1}}, \ldots, U_n^{p_{kn}}$ satisfy these relations. The validity of relations (1) and (2) for the elements $U_1^{p_{k1}}, \ldots, U_n^{p_{kn}}$ is obvious.

Let us check relation (3). It is clear that we have the operator inequality

$$U_1^{p_{k1}}\big(U_1^{p_{k1}}\big)^* + U_2^{p_{k2}}\big(U_2^{p_{k2}}\big)^* + \ldots + U_n^{p_{kn}}\big(U_n^{p_{kn}}\big)^* \le I. \tag{2.2}$$

According to the condition satisfied by the sequences (2.1), at least one of the exponents p_{kj} of the generating element U_j, $j = 1, \ldots, n$, in inequality (2.2) is greater than 1. Suppose that this is U_1. Moreover, suppose also that the equality

$$U_1^{p_{k1}}\big(U_1^{p_{k1}}\big)^* + U_2^{p_{k2}}\big(U_2^{p_{k2}}\big)^* + \ldots + U_n^{p_{kn}}\big(U_n^{p_{kn}}\big)^* = I \tag{2.3}$$

holds. Multiplying equality (2.3) by $(U_1^{p_{k1}-1})^*$ and $U_1^{p_{k1}-1}$ on the left and right, respectively, we obtain the equality

$$U_1 U_1^* = I,$$

which contradicts relation (3) for the generating elements $U_1, \ldots, U_n$ of the Toeplitz–Cuntz algebra $\mathcal{TO}_n$. Thus, we have shown the validity of relation (3) for the elements $U_1^{p_{k1}}, \ldots, U_n^{p_{kn}}$. $\square$

Consider the inductive sequence of Toeplitz–Cuntz algebras

$$\mathcal{TO}_n \xrightarrow{\varphi_1} \mathcal{TO}_n \xrightarrow{\varphi_2} \mathcal{TO}_n \xrightarrow{\varphi_3} \ldots \tag{2.4}$$

with connecting $*$-homomorphisms $\varphi_k \colon \mathcal{TO}_n \to \mathcal{TO}_n$ defined by the formulas from Lemma 1. We call this sequence the *inductive sequence of Toeplitz–Cuntz algebras defined by the set of sequences of positive integers* (2.1).

Below we will consider additive semigroups of rational numbers that correspond to the sequences (2.1) and are defined as

$$\mathbb{Q}^+_{P_k} = \left\{ \frac{m}{p_{1k} \cdots p_{sk}} \;\middle|\; m \in \mathbb{N},\ s \in \mathbb{N} \right\},$$

where $1 \le k \le n$. Introduce a semigroup S_k as the Cartesian product of n semigroups:

$$S_k = \{0\} \times \ldots \times \{0\} \times \mathbb{Q}^+_{P_k} \times \{0\} \times \ldots \times \{0\},$$

where the semigroup $\mathbb{Q}^+_{P_k}$ is in the kth position. Since $0 \notin \mathbb{Q}^+_{P_k}$ for each k, $1 \le k \le n$, it follows that $S_k \cap S_l = \varnothing$ for any k and l with $1 \le k, l \le n$. Let S be the unitalization of the free product of semigroups $S_1 * \ldots * S_n$. In other words, S is a semigroup obtained by adding the neutral element 0 to the semigroup $S_1 * \ldots * S_n$:

$$S = S_1 * \ldots * S_n \sqcup \{0\}.$$

It is easy to see that S is a cancellative semigroup. We will call S the *semigroup defined by the set of sequences of positive integers* (2.1).

The main result of this section is the fact that the inductive limit of the inductive sequence (2.4) coincides, up to isomorphism, with the reduced semigroup C^*-algebra $C^*_{\mathrm{r}}(S)$. To prove this fact, we consider another inductive sequence which is an inductive sequence of C^*-subalgebras of the semigroup C^*-algebra $C^*_{\mathrm{r}}(S)$.

Let us fix arbitrary elements $(q_1, 0, \dots, 0), \dots, (0, \dots, 0, q_n)$ in the free product of semigroups $S_1 * \dots * S_n$. Denote by $\mathcal{T}_{q_1,\dots,q_n}$ the C^*-subalgebra in $C^*_{\mathrm{r}}(S)$ generated by the n isometries $V_{(q_1,0,\dots,0)}, \dots, V_{(0,\dots,0,q_n)}$.

The following lemma shows that each of the algebras $\mathcal{T}_{q_1,\dots,q_n}$ is a universal C^*-algebra generated by n isometries with mutually orthogonal images, i.e., a Toeplitz–Cuntz algebra.

Lemma 2. *For any set of rational numbers $(q_1, \dots, q_n) \in \mathbb{Q}^+_{P_1} \times \dots \times \mathbb{Q}^+_{P_n}$, the correspondence $V_{(q_1,0,\dots,0)} \mapsto U_1, \dots, V_{(0,\dots,0,q_n)} \mapsto U_n$ extends to a C^*-algebra isomorphism*

$$\Psi\colon \mathcal{T}_{q_1,\dots,q_n} \to \mathcal{TO}_n.$$

Proof. Let us show that the elements $V_{(q_1,0,\dots,0)}, \dots, V_{(0,\dots,0,q_n)}$ generating the C^*-algebra $\mathcal{T}_{q_1,\dots,q_n}$ satisfy relations (1)–(3) from the definition of the Toeplitz–Cuntz algebra.

Relation (1) is obvious.

Let us prove relation (2). Fix k and l with $k \neq l$. Let us show that

$$V^*_{(0,\dots,q_k,\dots,0)} V_{(0,\dots,q_l,\dots,0)} e_a = 0$$

for any $e_a \in l^2(S)$. Suppose that $V^*_{(0,\dots,q_k,\dots,0)} V_{(0,\dots,q_l,\dots,0)} e_a \neq 0$. Then there exists an element $b \in S$ such that $V^*_{(0,\dots,q_k,\dots,0)} V_{(0,\dots,q_l,\dots,0)} e_a = e_b$. For the scalar product $\langle V^*_{(0,\dots,q_k,\dots,0)} V_{(0,\dots,q_l,\dots,0)} e_a, e_b\rangle$, we have

$$\begin{aligned}\big\langle V^*_{(0,\dots,q_k,\dots,0)} V_{(0,\dots,q_l,\dots,0)} e_a, e_b\big\rangle &= \big\langle V_{(0,\dots,q_l,\dots,0)} e_a, V_{(0,\dots,q_k,\dots,0)} e_b\big\rangle \\ &= \big\langle e_{(0,\dots,q_l,\dots,0)a}, e_{(0,\dots,q_k,\dots,0)b}\big\rangle = 0,\end{aligned}$$

since $(0, \dots, q_l, \dots, 0)a \neq (0, \dots, q_k, \dots, 0)b$. We have obtained a contradiction, since $\langle e_b, e_b\rangle = 1$. Thus, relation (2) holds.

It remains to prove relation (3). Obviously, we have the operator inequality

$$V_{(q_1,0,\dots,0)} V^*_{(q_1,0,\dots,0)} + V_{(0,q_2,0,\dots,0)} V^*_{(0,q_2,0,\dots,0)} + \dots + V_{(0,\dots,0,q_n)} V^*_{(0,\dots,0,q_n)} \leq I.$$

Suppose that

$$V_{(q_1,0,\dots,0)} V^*_{(q_1,0,\dots,0)} + V_{(0,q_2,0,\dots,0)} V^*_{(0,q_2,0,\dots,0)} + \dots + V_{(0,\dots,0,q_n)} V^*_{(0,\dots,0,q_n)} = I. \tag{2.5}$$

Take a number $q_0 \in \mathbb{Q}^+_{P_1}$ such that $q_0 < q_1$. Then $q_1 - q_0 \in \mathbb{Q}^+_{P_1}$. Multiplying equality (2.5) by $V^*_{(q_0,0,\dots,0)}$ and $V_{(q_0,0,\dots,0)}$ on the left and right, respectively, we obtain the operator equalities

$$V^*_{(q_0,0,\dots,0)} V_{(q_1,0,\dots,0)} V^*_{(q_1,0,\dots,0)} V_{(q_0,0,\dots,0)} = V^*_{(q_0,0,\dots,0)} V_{(q_0,0,\dots,0)} = I,$$

$$V^*_{(q_0,0,\dots,0)} V_{(q_0,0,\dots,0)} V_{(q_1-q_0,0,\dots,0)} V^*_{(q_1-q_0,0,\dots,0)} V^*_{(q_0,0,\dots,0)} V_{(q_0,0,\dots,0)} = I,$$

$$V_{(q_1-q_0,0,\dots,0)} V^*_{(q_1-q_0,0,\dots,0)} = I.$$

We have arrived at a contradiction, since the operator $V_{(q_1-q_0,0,\dots,0)}$ is not unitary.

Thus, the following strict inequality holds:

$$V_{(q_1,0,\dots,0)} V^*_{(q_1,0,\dots,0)} + V_{(0,q_2,0,\dots,0)} V^*_{(0,q_2,0,\dots,0)} + \dots + V_{(0,\dots,0,q_n)} V^*_{(0,\dots,0,q_n)} < I.$$

The existence of a required isomorphism of C^*-algebras $\Psi\colon \mathcal{T}_{q_1,\dots,q_n} \to \mathcal{TO}_n$ now follows from the Cuntz lemma [5, Lemma 3.1] (see also [16, Corollary 5.1]). $\square$

Now, let us construct the second inductive sequence of C^*-algebras. To this end, in the semigroup C^*-algebra $C^*_{\mathrm{r}}(S)$, we consider the sequence of unital C^*-subalgebras

$$\mathcal{T}_{1,\dots,1},\ \mathcal{T}_{\frac{1}{p_{11}},\dots,\frac{1}{p_{1n}}},\ \mathcal{T}_{\frac{1}{p_{11}p_{21}},\dots,\frac{1}{p_{1n}p_{2n}}},\ \dots,\ \mathcal{T}_{\frac{1}{p_{11}\cdots p_{k-1,1}},\dots,\frac{1}{p_{1n}\cdots p_{k-1,n}}},\ \dots.$$

Below, we will use the following notation for this sequence of algebras:

$$\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \dots, \mathcal{T}_k, \dots.$$

Denote the generating elements of the C^*-algebra $\mathcal{T}_1$ by

$$V_{11} := V_{(1,0,\dots,0)}, \quad \dots, \quad V_{1n} := V_{(0,\dots,0,1)}.$$

For the generating elements of the C^*-algebra $\mathcal{T}_k$, $k \geq 2$, we will also use the notation

$$V_{k1} := V_{\left(\frac{1}{p_{11}\cdots p_{k-1,1}},0,\dots,0\right)}, \quad \dots, \quad V_{kn} := V_{\left(0,\dots,0,\frac{1}{p_{1n}\cdots p_{k-1,n}}\right)}.$$

Note that the generators of the algebras $\mathcal{T}_k$ and $\mathcal{T}_{k+1}$, $k \in \mathbb{N}$, satisfy the equalities

$$V_{k1} = V_{k+1,1}^{p_{k1}}, \quad \dots, \quad V_{kn} = V_{k+1,n}^{p_{kn}}.$$

Hence, we have the inclusion of subalgebras $\mathcal{T}_k \subset \mathcal{T}_{k+1}$ for every positive integer k. Thus,

$$\mathcal{T}_1 \subset \mathcal{T}_2 \subset \mathcal{T}_3 \subset \dots \subset \mathcal{T}_k \subset \dots$$

is an increasing sequence of C^*-subalgebras of the semigroup C^*-algebra $C^*_{\mathrm{r}}(S)$, and

$$\mathcal{T}_1 \xrightarrow{\psi_1} \mathcal{T}_2 \xrightarrow{\psi_2} \mathcal{T}_3 \xrightarrow{\psi_3} \dots, \tag{2.6}$$

where the connecting $*$-homomorphisms ψ_k, $k \in \mathbb{N}$, are natural embeddings of C^*-subalgebras, is an inductive sequence of C^*-algebras. We also call the sequence (2.6) the *inductive sequence of C^*-subalgebras defined by the set of sequences of positive integers* (2.1).

In the following theorem, we prove that the closure of the union of the sequence of subalgebras $\{\mathcal{T}_k\}$ in the norm topology of the semigroup C^*-algebra $C^*_{\mathrm{r}}(S)$ coincides with the whole algebra $C^*_{\mathrm{r}}(S)$. This result implies (see, for example, [19, Remark 6.1.3]) that the inductive limit of the inductive sequence of C^*-subalgebras (2.6) is (up to isomorphism) the C^*-algebra $C^*_{\mathrm{r}}(S)$ itself.

Theorem 1. *Let S be the semigroup and $\{\mathcal{T}_k, \psi_k\}_{k=1}^{+\infty}$ the inductive sequence of C^*-subalgebras of the semigroup C^*-algebra $C^*_{\mathrm{r}}(S)$ that are defined by a set of sequences of positive integers $P_1, \dots, P_n$, $n \in \mathbb{N}$. Then the following equality holds*:

$$C^*_{\mathrm{r}}(S) = \overline{\bigcup_{k=1}^{+\infty} \mathcal{T}_k}. \tag{2.7}$$

Proof. The inclusion $C^*_{\mathrm{r}}(S) \supset \overline{\bigcup_{k=1}^{+\infty} \mathcal{T}_k}$ is obvious.

Let us show that the reverse inclusion holds. To this end, we prove the inclusion

$$P(S) \subset \bigcup_{k=1}^{+\infty} \mathcal{T}_k. \tag{2.8}$$

To prove this, we fix an arbitrary element $A \in P(S)$ and its representation in the form of a finite sum of operators of the form (1.1) with complex coefficients. In this linear combination, we consider an arbitrary term, which is assumed to have the form (1.1) without loss of generality. First, for each factor $V_{a_j}^{i_j}$ in this term (1.1), we represent the element a_j of the semigroup S as a sequence $a_j = b_1 \dots b_l$, $l \in \mathbb{N}$, of elements of the semigroups S_k; recall that any two adjacent elements in the sequence $b_1 \dots b_l$ belong to different semigroups S_k. Then we express the element $V_{a_j}^{i_j}$ as a composition of operators as follows:

$$V_{a_j}^{i_j} = \begin{cases} V_{b_1} V_{b_2} \dots V_{b_l} & \text{if } i_j = 1, \\ V_{b_l}^* V_{b_{l-1}}^* \dots V_{b_1}^* & \text{if } i_j = 0. \end{cases} \tag{2.9}$$

Suppose that we have the case $i_j = 1$ in (2.9). Let $b_1 \in S_k$ for some $1 \leq k \leq n$. Then there exist numbers $m \in \mathbb{N}$ and $s \in \mathbb{N}$ such that

$$b_1 = \left(0, \dots, 0, \frac{m}{p_{1k} \dots p_{sk}}, 0, \dots, 0\right),$$

where p_{jk} is the jth term of the sequence P_k, $1 \leq j \leq s$.

The operator equality

$$V_{b_1} = V_{s+1,k}^m$$

implies the inclusion $V_{b_1} \in \mathcal{T}_{s+1}$. In exactly the same way, for each index $i = 2, \dots, l$, one can show that the inclusion $V_{b_i} \in \mathcal{T}_{s_i}$ holds for some $s_i \in \mathbb{N}$. Therefore,

$$V_{a_j}^{i_j} \in \mathcal{T}_t, \tag{2.10}$$

where $t = \max\{s+1, s_2, \dots, s_l\}$.

In the case where the equality $i_j = 0$ holds in (2.9), inclusion (2.10) is proved in a similar way.

Thus, we have shown that each factor of the operator (1.1) is contained in the set on the right-hand side of (2.8). So we conclude that the operator (1.1) itself is contained in this set. Similarly, each term in the linear combination representing the operator A, and so the operator A itself, belongs to the right-hand side of (2.8). Thus, inclusion (2.8) is proved. Using (2.8) and the fact that the algebra $P(S)$ is dense in $C_{\mathrm{r}}^*(S)$, we conclude that equality (2.7) is valid. $\square$

Corollary 1. *Let S be the semigroup and $\{\mathcal{T}_k, \psi_k\}_{k=1}^{+\infty}$ the inductive sequence of C^*-subalgebras of the semigroup C^*-algebra $C_{\mathrm{r}}^*(S)$ that are defined by a set of sequences of positive integers $P_1, \dots, P_n$, $n \in \mathbb{N}$. Then the following C^*-algebra isomorphism holds*:

$$\varinjlim\{\mathcal{T}_k, \psi_k\} \cong C_{\mathrm{r}}^*(S).$$

Now we are ready to prove the main result of the paper on the limits of inductive sequences of Toeplitz–Cuntz algebras.

Theorem 2. *Let S be the semigroup and $\{\mathcal{TO}_n, \varphi_k\}_{k=1}^{+\infty}$ the inductive sequence of Toeplitz–Cuntz algebras that are defined by a set of sequences of positive integers $P_1, \dots, P_n$, $n \in \mathbb{N}$. Then the following C^*-algebra isomorphism holds*:

$$\varinjlim\{\mathcal{TO}_n, \varphi_k\} \cong C_{\mathrm{r}}^*(S).$$

Proof. By Lemma 2, for every $k \in \mathbb{N}$, there exists a C^*-algebra isomorphism $\Psi_k \colon \mathcal{T}_k \to \mathcal{TO}_n$. Denote by $\Phi_k \colon \mathcal{TO}_n \to \mathcal{T}_k$ the inverse of the isomorphism Ψ_k.

Consider the following diagram, in which the upper and lower rows are, respectively, the inductive Toeplitz–Cuntz sequence (2.4) and the inductive sequence of C^*-subalgebras of the semigroup C^*-algebra $C_{\mathrm{r}}^*(S)$ (2.6) that are defined by the sequences $P_1, \dots, P_n$:

$$\begin{array}{ccccccccccc}
\mathcal{TO}_n & \xrightarrow{\varphi_1} & \dots & \xrightarrow{\varphi_{k-1}} & \mathcal{TO}_n & \xrightarrow{\varphi_k} & \mathcal{TO}_n & \xrightarrow{\varphi_{k+1}} & \dots & & \varinjlim\{\mathcal{TO}_n, \varphi_k\} \\
\downarrow{\scriptstyle \Phi_1} & & & & \downarrow{\scriptstyle \Phi_k} & & \downarrow{\scriptstyle \Phi_{k+1}} & & & & \downarrow{\scriptstyle \varinjlim\{\Phi_k\}} \\
\mathcal{T}_1 & \xrightarrow{\psi_1} & \dots & \xrightarrow{\psi_{k-1}} & \mathcal{T}_k & \xrightarrow{\psi_k} & \mathcal{T}_{k+1} & \xrightarrow{\psi_{k+1}} & \dots & & \varinjlim\{\mathcal{T}_k, \psi_k\}
\end{array}$$

We argue that this diagram is commutative. Indeed, it suffices to show that for every $k \in \mathbb{N}$ the corresponding square is commutative, i.e., that the following equality of $*$-homomorphisms holds:

$$\Phi_{k+1} \circ \varphi_k = \psi_k \circ \Phi_k. \tag{2.11}$$

Let us show that equality (2.11) holds on the generating elements $U_1, \dots, U_n$ of the Toeplitz–Cuntz algebra $\mathcal{TO}_n$. We fix an arbitrary number $s \in \mathbb{N}$, $1 \le s \le n$, and first calculate the value of the homomorphism on the right-hand side of (2.11) on U_s:

$$\psi_k \circ \Phi_k(U_s) = \psi_k(V_{ks}) = V_{ks}.$$

Now we calculate the value of the homomorphism on the left-hand side of (2.11) on the same element U_s:

$$\Phi_{k+1} \circ \varphi_k(U_s) = \Phi_{k+1}(U_s^{p_{ks}}) = (\Phi_{k+1}(U_s))^{p_{ks}} = V_{k+1,s}^{p_{ks}} = V_{ks},$$

where p_{ks} is the kth term of the sequence P_s. Hence, on the generators of the Toeplitz–Cuntz algebra, equality (2.11) holds.

Next, recall (see the proof of Lemma 2) that the set of n operators

$$V_{k1}, \dots, V_{kn}$$

satisfies relations (1)–(3). Therefore, since the Toeplitz–Cuntz algebra $\mathcal{TO}_n$ is a universal C^*-algebra with n generators $U_1, \dots, U_n$ satisfying relations (1)–(3), there exists a unique $*$-homomorphism of C^*-algebras

$$\mathcal{TO}_n \to \mathcal{T}_{k+1}$$

that maps the generating element U_s to the operator V_{ks} for each $1 \le s \le n$. Hence, equality (2.11) holds, and so the diagram presented above is commutative.

Thus, the set $\{\Phi_k\}_{k=1}^{+\infty}$ is a morphism between the inductive sequences of C^*-algebras (2.4) and (2.6).

Finally, consider the C^*-algebra $*$-homomorphism

$$\varinjlim\{\Phi_k\}\colon\ \varinjlim\{\mathcal{TO}_n, \varphi_k\} \to \varinjlim\{\mathcal{T}_k, \psi_k\}$$

which is the limit morphism for the morphism $\{\Phi_k\}_{k=1}^{+\infty}$. Since all Φ_k are isomorphisms, the $*$-homomorphism $\varinjlim\{\Phi_k\}$ is also an isomorphism of C^*-algebras.

To complete the proof of the theorem, it remains to apply Corollary 1. □

In the final part of the paper, we address the problem of the existence of ideals in the semigroup C^*-algebra $C^*_{\mathrm{r}}(S)$, which is the inductive limit of the sequence of Toeplitz–Cuntz algebras defined by a set of sequences of positive integers. Namely, we show that the C^*-algebra $C^*_{\mathrm{r}}(S)$ is simple. It is well known that simple algebras play an important role in the structural theory of C^*-algebras.

Recall that a C^*-algebra is said to be *simple* if it has no closed ideals except the zero ideal and the algebra itself.

To prove the simplicity of the semigroup C^*-algebra $C^*_{\mathrm{r}}(S)$, we will apply the following well-known fact, which we formulate as a lemma.

Lemma 3. *Let $\mathcal{A}$ be a C^*-algebra containing an increasing sequence of C^*-subalgebras $\{\mathcal{A}_k\}_{k=1}^{+\infty}$ such that $\bigcup_{k=1}^{+\infty} \mathcal{A}_k$ is dense in $\mathcal{A}$. Let J be a closed ideal in $\mathcal{A}$. Then the following equality holds*:

$$J = \overline{\bigcup_{k=1}^{+\infty} (J \cap \mathcal{A}_k)}.$$

The proof of Lemma 3 is based on arguments similar to those used in the proof of Theorem 6.2.6 in [19] for approximately finite-dimensional algebras.

Let us proceed to the final statement of the paper, which states that the inductive limit of the inductive sequence of Toeplitz–Cuntz algebras defined by the set of sequences (2.1) is simple.

Theorem 3. *Let $\{\mathcal{TO}_n, \varphi_k\}_{k=1}^{+\infty}$ be the inductive sequence of Toeplitz–Cuntz algebras defined by the set of sequences of positive integers $P_1, \dots, P_n$, $n \in \mathbb{N}$. Then the inductive limit $\varinjlim\{\mathcal{TO}_n, \varphi_k\}$ of this sequence is a simple C^*-algebra.*

Proof. Let S be the semigroup defined by a set of sequences $P_1, \dots, P_n$. By Theorem 2, to prove the assertion, it suffices to prove that the C^*-algebra $C^*_{\mathrm{r}}(S)$ is simple.

Let J be a proper closed ideal of $C^*_{\mathrm{r}}(S)$.

We consider an increasing sequence of C^*-subalgebras $\{\mathcal{T}_k\}_{k=1}^{+\infty}$ in the algebra $C^*_{\mathrm{r}}(S)$ and first show that the equality

$$J \cap \mathcal{T}_k = \{0\} \tag{2.12}$$

holds for any positive integer k.

Suppose that $J \cap \mathcal{T}_k \neq \{0\}$ for some k. The set $J \cap \mathcal{T}_k$ is a closed ideal in the C^*-algebra $\mathcal{T}_k$. By Lemma 2, the algebra $\mathcal{T}_k$ is isomorphic to the Toeplitz–Cuntz C^*-algebra $\mathcal{TO}_n$. In [4, Proposition 3.1], Cuntz proved that the Toeplitz–Cuntz algebra $\mathcal{TO}_n$ contains a closed ideal K generated by the element $I - \sum_{k=1}^{n} U_k U_k^*$, which is isomorphic to the ideal of compact operators on an infinite-dimensional separable Hilbert space, and that the C^*-algebra isomorphism $\mathcal{TO}_n / K \cong \mathcal{O}_n$ holds, where $\mathcal{O}_n$ is the Cuntz algebra. Since the Cuntz algebra is simple [4, Theorems 1.12, 1.13], any proper nonzero ideal in $\mathcal{TO}_n$ coincides with K. Thus, in the C^*-algebra $\mathcal{T}_k$, there is an ideal generated by an element $I - \sum_{i=1}^{n} V_{ki} V_{ki}^*$ that coincides with the ideal $J \cap \mathcal{T}_k$. Hence, the following inclusion holds:

$$I - \sum_{i=1}^{n} V_{ki} V_{ki}^* \in J.$$

Let j and l, $1 \le j, l \le n$, be such that $j \neq l$ and $p_{kj} \neq 1$. Here we use the notation from (2.1) for the terms of the sequences $P_1, \dots, P_n$. Then we have the chain of equalities

$$\begin{aligned} V^*_{k+1,l} V^*_{k+1,j} \left(I - \sum_{i=1}^{n} V_{ki} V_{ki}^* \right) V_{k+1,j} V_{k+1,l} &= V^*_{k+1,l} V^*_{k+1,j} \left(I - \sum_{i=1}^{n} V^{p_{ki}}_{k+1,i} (V^*_{k+1,i})^{p_{ki}} \right) V_{k+1,j} V_{k+1,l} \\ &= V^*_{k+1,l} \big(I - V^*_{k+1,j} V^{p_{kj}}_{k+1,j} (V^*_{k+1,j})^{p_{kj}} V_{k+1,j} \big) V_{k+1,l} \\ &= V^*_{k+1,l} \big(I - V^{p_{kj}-1}_{k+1,j} (V^*_{k+1,j})^{p_{kj}-1} \big) V_{k+1,l} \\ &= I - V^*_{k+1,l} V^{p_{kj}-1}_{k+1,j} (V^*_{k+1,j})^{p_{kj}-1} V_{k+1,l} = I. \end{aligned}$$

Therefore, the identity operator I belongs to J, and the ideal J coincides with the whole C^*-algebra $C^*_{\mathrm{r}}(S)$. We have obtained a contradiction. Hence, equality (2.12) holds for any $k \in \mathbb{N}$.

Next, applying Theorem 1 and Lemma 3, we obtain the equality

$$J = \overline{\bigcup_{k=1}^{+\infty} (J \cap \mathcal{T}_k)}.$$

This means that $J = \{0\}$. The theorem is proved. $\square$

FUNDING

This work was performed in the framework of the development program of the Volga Region Scientific–Educational Centre of Mathematics (contract no. 075-02-2020-1478).

REFERENCES

1. W. Arveson, *Continuous Analogues of Fock Space* (Am. Math. Soc., Providence, RI, 1989), Mem. AMS **80** (409).
2. M. A. Aukhadiev, S. A. Grigoryan, and E. V. Lipacheva, "An operator approach to quantization of semigroups," Sb. Math. **205** (3), 319–342 (2014) [transl. from Mat. Sb. **205** (3), 15–40 (2014)].
3. O. Bratteli, P. E. T. Jorgensen, and G. L. Price, "Endomorphisms of $\mathcal{B}(\mathcal{H})$," in *Quantization, Nonlinear Partial Differential Equations, and Operator Algebra: 1994 John von Neumann Symposium on Quantization and Nonlinear Wave Equations, Cambridge, MA, 1994* (Am. Math. Soc., Providence, RI, 1996), Proc. Symp. Pure Math. **59**, pp. 93–138.
4. J. Cuntz, "Simple C^*-algebras generated by isometries," Commun. Math. Phys. **57**, 173–185 (1977).
5. J. Cuntz, "K-theory for certain C^*-algebras," Ann. Math., Ser. 2, **113** (1), 181–197 (1981).
6. N. J. Fowler, "States of Toeplitz–Cuntz algebras," J. Oper. Theory **42** (1), 121–144 (1999).
7. S. A. Grigoryan, E. V. Lipacheva, and A. S. Sitdikov, "Nets of graded C^*-algebras over partially ordered sets," St. Petersburg Math. J. **30** (6), 901–915 (2019) [transl. from Algebra Anal. **30** (6), 1–19 (2018)].
8. R. N. Gumerov, "Limit automorphisms of the C^*-algebras generated by isometric representations for semigroups of rationals," Sib. Math. J. **59** (1), 73–84 (2018) [transl. from Sib. Mat. Zh. **59** (1), 95–109 (2018)].
9. R. N. Gumerov, "Inductive limits for systems of Toeplitz algebras," Lobachevskii J. Math. **40** (4), 469–478 (2019).
10. R. N. Gumerov, "Inductive sequences of Toeplitz algebras and limit automorphisms," Lobachevskii J. Math. **41** (4), 637–643 (2020).
11. R. N. Gumerov and E. V. Lipacheva, "Inductive systems of C^*-algebras over posets: A survey," Lobachevskii J. Math. **41** (4), 644–654 (2020).
12. R. N. Gumerov, E. V. Lipacheva, and T. A. Grigoryan, "On inductive limits for systems of C^*-algebras," Russ. Math. **62** (7), 68–73 (2018) [transl. from Izv. Vyssh. Uchebn. Zaved., Mat., No. 7, 79–85 (2018)].
13. R. N. Gumerov, E. V. Lipacheva, and T. A. Grigoryan, "On a topology and limits for inductive systems of C^*-algebras over partially ordered sets," Int. J. Theor. Phys. **60** (2), 499–511 (2021).
14. J. M. Howie, *An Introduction to Semigroup Theory* (Academic, London, 1976).
15. M. Laca, "Endomorphisms of $\mathcal{B}(\mathcal{H})$ and Cuntz algebras," J. Oper. Theory **30** (1), 85–108 (1993).
16. M. Laca and I. Raeburn, "Semigroup crossed products and the Toeplitz algebras of nonabelian groups," J. Funct. Anal. **139** (2), 415–440 (1996).
17. E. V. Lipacheva, "Embedding semigroup C^*-algebras into inductive limits," Lobachevskii J. Math. **40** (5), 667–675 (2019).
18. E. V. Lipacheva and K. H. Hovsepyan, "Automorphisms of some subalgebras of the Toeplitz algebra," Sib. Mat. J. **57** (3), 525–531 (2016) [transl. from Sib. Mat. Zh. **57** (3), 666–674 (2016)].
19. G. J. Murphy, *C^*-Algebras and Operator Theory* (Academic, Boston, 1990).

Translated by I. Nikitin