Journal of Applied Mathematics and Mechanics 78 (2014) 163–171

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

The dynamic process of the inflation of thin elastomeric shells under the action of an excess pressure☆

P. Sh. Gimadiyev*, T.Z. Gimadiyeva, V.N. Paimushin

Kazan, Russia

ARTICLE INFO

Article history:
Received 5 August 2013

ABSTRACT

A problem of the dynamic process of their deformation is formulated in the momentless approximation for thin shells made of rubber-like elastomers under the action of a time-varying excess hydrostatic pressure. A system of non-linear equations of motion is set up for the case of arbitrary displacements and deformations in which the true deformation of the transverse compression of the shell, corresponding to the use of the modified Kirchhoff–Love model proposed earlier, and the coordinates of the points of the middle surface with respect to a fixed Cartesian system of coordinates, are taken as the required unknown functions. Physical relations connecting the components of the true internal stresses with the elongation factors and the extent of the shear strain are constructed using relations proposed earlier by Chernykh. A finite-difference method is developed for solving the initial-boundary value problem and, on the basis of this, the dynamic process of the inflation of shells of revolution at different rates of pressure increase is investigated and the unstable stages of their deformation are established with a determination of the corresponding limiting (critical) pressure value. After this value has been reached, a further increase in the deformations occurs at decreasing values of the internal pressure.

Products that are thin-walled shells made of highly elastic materials (a synthetic elastomer, latex film or natural rubber) and subjected to considerable deformations (a relative elongation of up to 1000%) during use are widely and diversely applied: catheters used in medicine, car air bags, air balloons, etc. As a rule, calculations of the strength of such constructional components must be based on the use of the relations of the non-linear mechanics of deformable solids and thin shells for finite displacements and deformations. There is an extensive literature [1–12] dealing with the construction of one version or another of these relations. Examples of their application in solving certain problems in the mechanics of elastomers have been presented, in particular, in a monograph.[3] The physical relations constructed in it, relating the components of the true stresses to the components of the true strains in the form of elongation factors have been used [11] to formulate and solve problem of the inflation and static stability of a cylindrical shell with closed ends made of rubber and under the action of an internal pressure. A characteristic feature of this problem is the separation of the process of loading the shell into two stages: in the first stage, an increase in the diameter and length of the shell only occurs when the pressure increases and, in the second stage, a further increase in the above dimensions of the shell and a decrease in its thickness occurs by pumping air into the shell with decreasing pressure. The mechanical explanation of this process involves the onset of the static instability of the rubber shell under conditions of biaxial asymmetric stretching, similar to the formation of a neck in cylindrical samples made of elastoplastic materials when they are stretched [10] in an axial direction under a static load.

The purpose of this paper is to study the loading of thin elastomer shells with an internal pressure described above within the limits of a dynamic formulation of the problem, that extends the results of the investigations carried out earlier.[4,7,10,11] It follows, starting out from an analysis of the results obtained earlier,[11] that taking account of the deformation of the compression of the shell in the transverse direction, the finiteness of the components of the true deformations, the introduction of the true stresses according to Novoshilov [1] and the use of constitutive relations linking the true stresses and true strains with one another is of fundamental importance in its formulation.

☆ *Prikl. Mat. Mekh.*, Vol. 78, No. 2, pp. 236–248, 2014.
* Corresponding author.
E-mail address: vpajmushin@mail.ru (V.N. Paimushin).

http://dx.doi.org/10.1016/j.jappmathmech.2014.07.009
0021-8928/

1. The equations of motion of a momentless shell

We assume that, at the instant $t = t_0$, the space V_0 of the undeformed shell is referred to a system of curvilinear coordinates α^1, α^2, z which is normally associated with the middle surface σ_0 that has the principal basis vectors $\mathbf{r}_i^0 = \partial \mathbf{r}^0/\partial\alpha^i$ and components of the principal metric tensor $G_{ij}^0 = \mathbf{r}_j^0\mathbf{r}_j^0$. In the system of coordinates adopted, the radius vector of an arbitrary point $M_0 \in V_0$ is defined by the equality (henceforth Latin indices have the values 1,2 and Greek indices have the values 1, 2, 3)

$$\mathbf{R}^0(\alpha^i, z) = \mathbf{r}^0(\alpha^i) + z\mathbf{e}_3^0, \quad -h/2 \le z \le h/2 \tag{1.1}$$

where $\mathbf{r}^0 = \mathbf{r}^0(\alpha^i)$ is the radius vector of a point on the surface σ_0, h is the initial thickness of the shell, and $\mathbf{e}_3^0$ is the vector of the unit normal to the surface σ_0 that constitute a right handed trihedron with the unit vectors $\mathbf{e}_i^0 = \mathbf{r}_i^0/\sqrt{G_{ii}^0}$.

When the shell is dynamically deformed, we shall define the radius vector of the above-mentioned point $M_0 \in V_0$, that has passed at the instant t to the point $M(\alpha^i, z) \in V$, according to the modified Kirchhoff–Love theory,[11] by the representation

$$\mathbf{R} = \mathbf{R}^0 + \mathbf{U} = \mathbf{r} + z(1+\varphi)\mathbf{e}_3, \quad \mathbf{U} = \mathbf{u} + z[(1+\varphi)\mathbf{e}_3 - \mathbf{e}_3^0] \tag{1.2}$$

where $\mathbf{u} = u^\alpha \mathbf{e}_\alpha^0$ is the vector of the displacements of the points of the middle surface σ_0, $\varphi(\alpha^i)$ is a transverse deformation function that is introduced into the treatment in terms of which the elongation factor λ_3 and the true strain ε_3 in the transverse direction z are defined using the formulae [11]

$$\lambda_3 = 1 + \varphi, \quad \varepsilon_3 = \varphi \tag{1.3}$$

and $\mathbf{e}_\alpha$ are unit vectors in the deformed surface σ with a radius vector $\mathbf{r}$, to determine which we have the formulae

$$\begin{gathered}\mathbf{e}_i = \mathbf{r}_i/A_i, \quad A_i = \sqrt{G_{ii}}, \quad \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2\sqrt{G_{11}G_{22}}/\sqrt{G} \\ G = G_{11}G_{22} - G_{12}^2, \quad G_{ij} = \mathbf{r}_i\mathbf{r}_j, \quad \mathbf{r}_i = \partial\mathbf{r}/\partial\alpha^i\end{gathered} \tag{1.4}$$

Note that the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are directed along the tangents to the coordinate lines α^i in the deformed state and $\mathbf{e}_3$ is directed along the normal to the surface σ. The covariant components of the strain tensor

$$\varepsilon_{ij} = (G_{ij} - G_{ij}^0)/2 \tag{1.5}$$

that serve for the calculating of the elongation factors λ_1 and λ_2 in the direction of the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ and the shear measure $\sin\gamma_{12}$ [9,11] in accordance with the expressions (φ_{12} is the angle between the basis vectors $\mathbf{r}_1^0$ and $\mathbf{r}_2^0$ in the undeformed state)

$$\lambda_i = 1 + \varepsilon_i = \sqrt{1 + 2\varepsilon_{(ii)}}, \quad \sin\gamma_{12} = \frac{2\varepsilon_{(12)}}{\lambda_1\lambda_2\sin\varphi_{12}}, \quad \varepsilon_{(ij)} = \frac{\varepsilon_{ij}}{\sqrt{G_{ii}^0G_{jj}^0}} \tag{1.6}$$

where ε_i is the relative elongation and $\varepsilon_{(ij)}$ is the dimensionless value of the covariant components of the strain tensor, are determined by the difference between the components of the metric tensors G_{ij} and G_{ij}^0.

Assuming that it is a thin momentless shell, in the sections $\alpha^i = \text{const}$ and $z = \text{const}$ of the deformed shell which, at the instant t, has a thickness [11]

$$h_* = h(1 + \varepsilon_3) = h(1 + \varphi) = h\lambda_3$$

we introduce the vectors of the true stress σ^i and σ_3 into the treatment, defining them by the representations

$$\sigma^i = \sigma^{ij}\mathbf{e}_j, \quad \sigma^3 = \sigma^{33}\mathbf{e}_3 \tag{1.7}$$

in which the quantities σ^{ij} and σ^{33} are physical components.

Integrating expression (1.7) over the thickness of the shell h_*, we obtain

$$\mathbf{T}^i = T^{ij}\mathbf{e}_j, \quad \mathbf{T}^3 = T^{33}\mathbf{e}_3 \tag{1.8}$$

where

$$T^{ij} = h\lambda_3\sigma^{ij}, \quad T^{33} = h\lambda_3\sigma^{33} \tag{1.9}$$

We will now assume that surface forces p^- and p^+, applied to points of the faces $z = -h_*/2$ and $z = h_*/2$, as well as a mass force $\mathbf{Q}$ act on an infinitesimal element of thickness h_* separated from the shell and on the surface σ, that is, an infinitesimal area $d\sigma = \sqrt{G}d\alpha^1 d\alpha^2$. We will assume that they are defined in the form

$$\mathbf{p} = p\mathbf{e}_3 d\sigma = p\mathbf{e}_3\sqrt{G}d\alpha^1 d\alpha^2, \quad \mathbf{Q} = \mathbf{g}h_*\rho dF = \mathbf{g}h_*\rho\sqrt{G}d\alpha^1 d\alpha^2 \tag{1.10}$$

where $p = p^- + p^+$ is the excess pressure acting on the shell divided by the unit of area $d\sigma$, ρ is the density of the shell material that we shall subsequently consider as invariable when treating shells made of an incompressible elastomer, and $\mathbf{g}$ is the gravitational acceleration.

In the approximation of momentless theory, the transverse internal stress formed in the shell T^{33} and the projection of the principal moment of the external forces in the direction of the normal $\mathbf{e}_3$ that, in the case considered, is equal to

$$M^3 = (p^+ - p^-)h_*$$

must, according to the results obtained earlier,[11] satisfy an equilibrium equation of the form

$$T^{33} = -(p^+ - p^-)h_* \tag{1.11}$$

By virtue of the fact that the estimates

$$\sigma^{11} \gg \sigma^{12}, \quad \sigma^{22} \gg \sigma^{12}, \quad \sigma^{11} \gg \sigma^{33}, \quad \sigma^{22} \gg \sigma^{33}, \quad \sigma^{33} \sim (p^- - p^+)$$

hold for the true stresses σ^{11}, σ^{12}, σ^{22}, σ^{33} introduced into the treatment, equality (1.1) can be replaced by the equality $T^{33} \approx 0$ which enables us to assume that the formed stress-strain state (SSS) of the shell is a plane stressed state. Applying this to the infinitesimal elements separated from the shell, all the above-mentioned internal and external stresses as well as the inertial forces can constitute, starting out from the d'Alembert principle in vector form, an equation of motion of the following form:

$$\rho h_* \sqrt{G}\frac{\partial^2 \mathbf{r}}{\partial t^2} = \frac{\partial}{\partial \alpha^1}((T^{11}\mathbf{e}_1 + T^{12}\mathbf{e}_2)\sqrt{G_{22}}) + \frac{\partial}{\partial \alpha^2}((T^{22}\mathbf{e}_2 + T^{21}\mathbf{e}_1)\sqrt{G_{11}}) + (\mathbf{p} + \rho h_* \boldsymbol{g})\sqrt{G} \tag{1.12}$$

in which, unlike in the equations obtained earlier,[11] the vector function $\mathbf{r}$ and the components of the true internal stresses T^{ij} are unknowns.

From here on, instead of the three unknown displacement functions $u^\gamma(\alpha^1, \alpha^2, t)$, we introduce new unknowns into the treatment, taking the representation

$$\mathbf{r} = \mathbf{r}(\alpha^1, \alpha^2, t) = \sum_{\gamma=1}^{3} x_\gamma \mathbf{i}_\gamma = x_\gamma \mathbf{i}_\gamma, \quad x_\gamma = x_\gamma(\alpha^1, \alpha^2, t) \tag{1.13}$$

for $\mathbf{r}$, where x_γ ($\gamma = 1, 2, 3$) are new unknowns that are the coordinates of an arbitrary point on σ with respect to a fixed orthogonal Cartesian system of coordinates $Ox_1x_2x_3$ with unit vectors $\mathbf{i}_\gamma$. To determine the principle basis vectors $\mathbf{r}_i$ and the components of the metric tensor G_{ij} we shall use the relations

$$\mathbf{r}_j = \partial \mathbf{r}/\partial \alpha^j = r_{j,\gamma}\mathbf{i}_\gamma, \quad r_{j,\gamma} = \partial x_\gamma / \partial \alpha^j, \quad G_{jm} = \mathbf{r}_j \mathbf{r}_m = r_{j,k} r_{m,k} \tag{1.14}$$

In this case,

$$\mathbf{e}_i = \mathbf{r}_i/\sqrt{G_{ii}} = l_{i\gamma}\,\mathbf{i}_\gamma, \quad \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2\sqrt{G_{11}G_{12}}/\sqrt{G} \tag{1.15}$$

where $l_{\delta\gamma} = \cos(\mathbf{e}_\delta, \mathbf{i}_\gamma)$ are direction cosines defined by the expressions

$$\begin{aligned} l_{j\gamma} &= r_{j,\gamma}/\sqrt{G_{jj}} \\ l_{31} &= (l_{12}l_{23} - l_{13}l_{22})\sqrt{G_{11}G_{22}}/\sqrt{G} \\ l_{32} &= (l_{13}l_{21} - l_{11}l_{23})\sqrt{G_{11}G_{22}}/\sqrt{G} \\ l_{33} &= (l_{11}l_{22} - l_{12}l_{21})\sqrt{G_{11}G_{22}}/\sqrt{G} \end{aligned} \tag{1.16}$$

In constructing the scalar equations of motion in a Cartesian system of coordinates for the vector $\mathbf{g}$, we will assume that the representation $\mathbf{g} = g\mathbf{i}_3$ holds. Then, by projecting Eq. (1.12) onto the Cartesian axes and taking account of relation (1.16), we obtain the three equations of motion

$$\begin{aligned} \rho h_* \sqrt{G}\frac{\partial^2 x_\gamma}{\partial t^2} &= \frac{\partial}{\partial \alpha^1}((T^{11}l_{1\gamma} + T^{12}l_{2\gamma})\sqrt{G_{22}}) \\ &+ \frac{\partial}{\partial \alpha^2}((T^{22}l_{2\gamma} + T^{21}l_{1\gamma})\sqrt{G_{11}}) + p_3 l_{3\gamma}\sqrt{G}(1 - \delta_{3\gamma}\rho h_* g) \end{aligned} \tag{1.17}$$

where $\delta_{31} = \delta_{32} = 0$, $\delta_{33} = 1$, $\lambda = 1, 2, 3$.

For the closure of the system of equation obtained, consisting of the equations of motion (1.17), the geometric and kinematic relations (1.14) and (1.16), formulae (1.8) and the equality $T^{33} \approx 0$, it is necessary to construct physical relations interconnecting the stresses σ^{ij} and σ^{33} with the deformation parameters in the form of the elongation factors $\lambda_i = 1 + \varepsilon_i$, $\lambda_3 = 1 + \varepsilon_3 = 1 + \varphi$ and the shear measure $\sin\gamma_{12}$. By virtue of the fact that

$$h_* = h(1 + \varphi), \quad \sqrt{G_{ii}} = A_i = A_i^0(1 + \varepsilon_i) = A_i^0\lambda_i$$

where $A_i^0 = \sqrt{G_{ii}^0}$, the stresses σ^{1j} in the resulting equations, which are true stresses, contain the factors $\lambda_3\lambda_2$ and the stresses σ^{2j} contain the factors $\lambda_3\lambda_1$. Hence, in view of the results obtained earlier [9,11] for the version of system on equations (1.17) considered, the physical relations must have the following structure:

$$\sigma^{ij} = \sigma^{ij}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \sin\gamma_{12}), \quad \sigma^{33} = \sigma^{33}(\varepsilon_1, \varepsilon_2, \varepsilon_3) \tag{1.18}$$

2. Thin elastomer shells of revolution under the action of an internal pressure

When $h = h(\alpha^1)$, these shells can only be in an axisymmetric stress-strain state if the force of gravity $\rho h_* g\sqrt{G}$ is not taken into account in the last equation of motion (1.17) ($\gamma = 3$). With respect to their mechanical behaviour, they are close to the soft fabric shells,[5–7] that have been studied in detail in parachute construction [9] and in structural pneumatic and marine constructions [13,14] that, in an "operational" state, can be under conditions of bilateral tension. With the appearance of stresses σ^{12}, when $\sigma^{11} > 0$, $\sigma^{22} > 0$ or under conditions when $\sigma^{11} > 0$ for $\sigma^{22} \le 0$ and $\sigma^{22} > 0$ for $\sigma^{11} \le 0$, local zones of wrinkling of the shell appear in them and, in this case, the shell changes into a state of uniaxial loading.

We shall henceforth assume that a stress-strain state is formed in the shells considered, which satisfies the conditions and estimates

$$\sigma^{12} \approx 0, \quad \sigma^{11} > 0, \quad \sigma^{22} > 0, \quad \sigma^{11}, \sigma^{22} \gg \sigma^{12} \tag{2.1}$$

By virtue of conditions and estimates (2.1), instead of (1.18) we shall use the physical relations

$$\sigma^{ii} = \sigma^{ii}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \dot{\varepsilon}_1, \dot{\varepsilon}_2), \quad \sigma^{33} = \sigma^{33}(\varepsilon_1, \varepsilon_2, \varepsilon_3), \quad \dot{\varepsilon}_i = \partial \varepsilon_i / \partial t$$

introducing terms into them that take account of internal friction in the shell material in order to describe rapid deformation processes with considerable rates of strain $\dot{\varepsilon}_i$. In the case of rubber-like compressible materials, these relations can be constructed starting from Chernykh's results [3] and the Kelvin–Voigt model that, by virtue of the adopted assumption $\sigma^{33} \approx 0$, can be represented in the form

$$\begin{aligned} &\sigma^{ii} \approx \mu[(1+\beta)\lambda_i - (1-\beta)\lambda_i^{-1}] + q + \eta\dot{\varepsilon}_i \\ &\mu[(1+\beta)\lambda_3 - (1-\beta)\lambda_3^{-1}] + q = 0 \end{aligned} \tag{2.2}$$

where q is the hydrostatic pressure, μ and β are characteristics of the material that, for one of the rubbers (IRP-2052), are equal to $\mu = 8.75$ and $\beta = 1.43$ [3] and the characteristic μ corresponds in a physical sense to a modulus of elasticity of the first kind E and η is a parameter characterizing the internal viscous friction and has the sense of a complex modulus of elasticity.[15] It should be noted that, in rapidly occurring dynamic processes, by taking account of internal friction in relations (2.2), we can damp the velocities of the stress waves in the α^1 and α^2 directions and ensure the stability of the numerical algorithms for solving the problems considered below.

After eliminating the unknown function q from relations (2.2), physical relations of the following form

$$T^{ii} = h_*\{\mu[(1+\beta)(\lambda_i - \lambda_1^{-1}\lambda_2^{-1}) + (1-\beta)(\lambda_1\lambda_2 - \lambda_i^{-1})] + \eta\dot{\varepsilon}_i\} \tag{2.3}$$

are established in accordance with known results [3,9] and formulae (1.9).

When $\mathbf{r}_1^0\mathbf{r}_2^0 = 0$ (orthogonal curvilinear coordinates) and account is taken of the stresses σ^{12}, relations (2.3) can be supplemented with the relation

$$T^{12} = T^{21} = h_*\left(K_{12}\frac{G_{12}}{\lambda_1\lambda_2} + \eta_\gamma \dot{G}_{12}\right) \tag{2.4}$$

in which η_γ is the internal viscous friction shear strain parameter, G_{12} is calculated using formula (1.14) and $K_{12} = K_{12}(G_{12})$ is the tangential shear modulus of the shell material, and its dependence on the shear strain measure G_{12} (or [3,11] $\sin\gamma_{12}$) can only be established experimentally.

In solving specific problems on the inflation of a shell of revolution, as it is blown up we approximate the excess pressure p acting at points of the face $z = -h_*/2$ with the relation

$$p(t) = \tilde{p}(t)(1 - \nu^n V^n)^2 \operatorname{sgn}(1 - \nu^n V^n) \tag{2.5}$$

that is widely used in the theory of soft shells when studying parachute dynamics.[7] In this relation, the law for the distribution of the excess pressure $\tilde{p}(t)$ with respect to the spatial coordinates α^1 and α^2 is assumed to be homogeneous and given. Terms are also introduced in it that take account of the pressure change from the oscillations of the shell itself: when an element of the shell moves against the internal pressure acting on this element the pressure drop increases and, when it moves in the opposite direction, it decreases, that is, there is a damping of the bulging by the oscillation of the shell itself. With the aim of taking account of this dynamic process, the term $\nu^n V^n$ has been introduced into relation (2.5) and, in this term, ν^n is the aerodynamic damping factor of a medium when a shell moves in it with a velocity $V^n = \partial u_3/\partial t$ in the direction of the normal $\mathbf{e}_3$ to the surface σ. The component u_3 of the displacement vector is calculated using the formula $u_3 = \mathbf{u}\mathbf{e}_3$.

We shall henceforth assume than, when $t = t_0$, the initial shape of the shell σ_0 is described by the equation

$$\mathbf{r}(\alpha^1, \alpha^2, t_0) = \mathbf{r}_0(\alpha^1, \alpha^2)$$

in which α^1 is a curvilinear coordinate along the generatrix of the shell that varies within the limits $l_0 \le \alpha^1 \le l_1$ and the circumferential coordinate α^2, that is, the polar angle that varies within the limits $0 \le \alpha^2 \le 2\pi$. We also assume that the elements of the shell are stationary at the initial instant:

$$\left.\frac{d\mathbf{r}(\alpha^1, \alpha^2, t)}{dt}\right|_{t=t_0} = \mathbf{V}_0(\alpha^1, \alpha^2) = 0$$

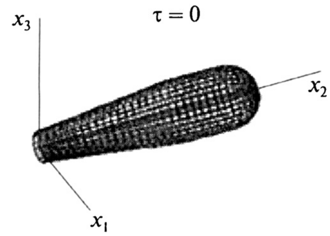

Fig. 1.

We shall assume that the boundary conditions are given for the equations of motion that have been constructed, representing the equation of the surface σ, when $\alpha^1 = l_0$ and $t \geq t_0$, in the form

$$\mathbf{r}(l_0, \alpha^2, t) = \mathbf{r}_0(\alpha^1 = l_0, \alpha^2)$$

The inlet aperture of the shell (Fig. 1) is assumed to be fixed and the remaining parts of the shell are displaced in space.

3. A difference scheme for solving the problem

Consider an element (i, j) of the difference mesh covering the deformed surface and let us assume that its mass is concentrated at the mesh point (i, j).

We shall calculate the stretching factors in the directions of α^1 and α^2 at each instant using the formulae

$$\lambda_1^{i+1/2,j} = h_1^{-1}\left(\sum_{j=1}^{3}(x^{\gamma}_{i+1,j} - x^{\gamma}_{i,j})^2\right)^{1/2}$$

$$\lambda_2^{i,j+1/2} = h_2^{-1}\left(\sum_{j=1}^{3}(x^{\gamma}_{i,j+1} - x^{\gamma}_{i,j})^2\right)^{1/2}$$

$$\lambda_1^{i+1/2,j+1/2} = (\lambda_1^{i+1/2,j+1} + \lambda_1^{i+1/2,j})/2$$

$$\lambda_2^{i+1/2,j+1/2} = (\lambda_2^{i+1,j+1/2} + \lambda_2^{i,j+1/2})/2 \tag{3.1}$$

where $h_1 = h_1(i, j)(h_2 = h_2(i, j))$ is the distance between the mesh points $(i + 1/2, j)((i, j + 1/2))$ and (i,j).

The relations

$$\{l_{1,\gamma}\}\Big|_j^i = \frac{x^{\gamma}_{i+1,j} - x^{\gamma}_{i,j}}{h_1\lambda_1^{i+1/2,j}}, \quad \{l_{2,\gamma}\}\Big|_i^j = \frac{x^{\gamma}_{i,j+1} - x^{\gamma}_{i,j}}{h_2\lambda_2^{i,j+1/2}} \tag{3.2}$$

are used to calculate the direction cosines of the principle basis vectors.

We match their difference analogues to the equations of motion (1.17) that allow of the representation with respect to the velocity vector components in a certain generalized form (A and B are certain constants)

$$\partial V/\partial t - A\partial T/\partial s = B \tag{3.3}$$

For example, when $2 \leq i \leq n_1 - 1$, $2 \leq j \leq n_2 - 1$,

$$\begin{aligned}
\{V^{\gamma}\}_{i,j}^{n+1/2} = \{V^{\gamma}\}_{i,j}^{n-1/2} + \frac{\Delta t}{\rho h_*}\Bigg(&\sum_{\pm}[\{T^{11}\lambda_2\}_{j\pm1/2}^{i+1/2}\{l_{1\gamma}\}^{i}_{j\pm1/2} - \{T^{11}\lambda_2\}_{j\pm1/2}^{i-1/2}\{l_{1\gamma}\}^{i}_{j\pm1/2} \\
&+ \{T^{12}\lambda_2\}_{i+1/2}^{j\pm1/2}\{l_{2\gamma}\}^{j\pm1/2}_{i+1/2} - \{T^{12}\lambda_2\}_{i-1/2}^{j\pm1/2}\{l_{2\gamma}\}^{j\pm1/2}_{i-1/2}]/(2h_1) \\
&+ \sum_{\pm}[\{T^{22}\lambda_1\}_{i\pm1/2}^{j+1/2}\{l_{2\gamma}\}^{j+1/2}_{i\pm1/2} - \{T^{22}\lambda_1\}_{i\pm1/2}^{j-1/2}\{l_{2\gamma}\}^{j-1/2}_{i\pm1/2} \\
&+ \{T^{21}\lambda_1\}_{j+1/2}^{i\pm1/2}\{l_{1\gamma}\}^{i\pm1/2}_{j+1/2} - \{T^{21}\lambda_1\}_{j-1/2}^{i\pm1/2}\{l_{1\gamma}\}^{i\pm1/2}_{j-1/2}]/(2h_2) + \rho h_*(1 - \delta_{3\gamma})\Bigg) + P_{i,j}\frac{\Delta t}{h_1h_2}F^{\gamma}_{i,j}
\end{aligned} \tag{3.4}$$

Here, by way of illustration,

$$\{l_{1\gamma}\}^{i}_{j-1/2} = [x^{\gamma}_{i+1,j-1/2} - x^{\gamma}_{i,j-1/2}]/[h_1\lambda^1_{i+1/2,j-1/2}]$$

and $F^{\gamma}_{i,j}$ is the sum of the projections of the area of the eight triangles adjacent to the mesh point (i,j) in the plane $x^{\gamma}=0(\gamma=1, 2, 3)$. Thus, for the first triangle we have

$$F^1_{i,j}=[x^2_{i,j}(x^3_{i+1/2,j}-x^3_{i+1/2,j+1/2})+x^2_{i+1/2,j}(x^3_{i+1/2,j+1/2}-x^3_{i,j})\\+x^2_{i+1/2,j+1/2}(x^3_{i,j}-x^3_{i+1/2,j})]/2 \tag{3.5}$$

The projection of the area onto the other planes $x^2=0$ and $x^3=0$ are determined in a similar way. In this case, the indices 1, 2, 3 change as a cyclic permutation.

The integration step is chosen in accordance with the Courant–Friedrichs–Lewy criterion

$$\Delta t<\alpha_k\min(h_1,h_2)/c \tag{3.6}$$

where α_k is the Courant coefficient and c is the speed of propagation of small perturbations in the material (the speed of sound).

3.1. *Estimation of the error in approximating of the derivatives*

In Eq. (3.3), consider the derivative $\partial T/\partial s$ with respect to the half-integer mesh

$$s_{i+1/2}=(i+1/2)h,\qquad i=0,\ 1,\ 2,\ldots,\ M-1$$

at a fixed time $t_n(n=0, 1, 2, \ldots)$. For this derivative, by expanding the values of the function T at the half-integral points in Taylor series, in the neighbourhood of an integral point one can write [16]

$$\frac{T(s_{i+1/2},t_n)-T(s_{i-1/2},t_n)}{h}=\frac{\partial T}{\partial s}(s_i,t_n)+\frac{h^2}{24}\frac{\partial^3 T}{\partial s^3}(s_i,t_n)+O(h^4)$$

whence it follows that the central difference with respect to the Lagrangian coordinate on the half-integral mesh approximates the derivative $\partial T/\partial s$ with second order of accuracy.

The derivatives $\partial V/\partial t$ are calculated in a similar way and the values of Δt and Δs are connected by linear relation (3.6), $\Delta t=c_1h$, where $c_1=\alpha_k/c$. Consequently, the central difference with respect to the half-integral mesh approximate the partial derivatives $\partial T/\partial s$, $\partial V/\partial t$ in Eq. (3.3) with second order of accuracy.

We shall satisfy the boundary conditions for the shell on an expanded mesh, the dimensions of which are determined by the numbers 1, n_1 and 1, n_2. In this case, the indices i and j will vary within the limits $1\le i\le n_1$, $1\le j\le n_2$. The required coordinates of the mesh points of the shell in the time layers are determined using the formulae

$$\{x^{\gamma}\}^{n+1}_{i,j}=\{x^{\gamma}\}^{n}_{i,j}+\Delta t\{\tilde V^{\gamma}\}^{n+1/2}_{i,j},\qquad \gamma=1,2,3 \tag{3.7}$$

Hence, an explicit finite difference scheme is used as the basis for constructing the numerical solution of the problem. As a result, when constructing the solution of the system of equations that has been compiled, the discrete domain

$$S_{n_i}=n_ih_i,\quad n_i=1,2,3,\ldots,\quad S_i/\Delta s_i;\quad t_n=n\Delta t\ ,\quad n=0,1,2,\ldots,\quad t/\Delta t-1$$

is introduced into the treatment. Here, we determine the values of the required functions at each integration step in terms of the their already known values at the preceding step in the framework of a single continuous computational algorithm.

At the initial instant when $n=0$ (that is, when $t=-\Delta t/2$), in equations of the form (3.4), it is necessary to specify $\left\{V^{\gamma}\right\}^{-1/2}_{i,j}$ on the expanded mesh for the start of the calculation. Thereafter, for a time $t>0$, these velocities are recalculated on the expanded mesh at each step.

The choice of the coefficients η and η_{γ} (formulae (2.3) and (2.4)), that take account of the internal friction in the material on the dynamics of the shell, the integration step size Δt and the computational mesh steps h_1 and h_2 associated with relation (3.6), is obtained from a numerical experiment.

Note that the algorithm described also enables one to study static problems of the mechanics of elastomeric shells by the establishment method.

4. The equations of motion in dimensionless form

To represent Eq. (1.12) in dimensionless form, the following are given: the characteristic size L (m), that is, the length of the generatrix of the shell, the pressure drop and the density of air ρ_B (kg/m^3), by using which the characteristic velocity V_0 (m/s) is determined, and the characteristic force $T_0=\rho_BV_0^2L^2$[H] and the characteristic mass $M_s=2\pi rL_\rho$ are introduced into the treatment, where r is the maximum radius of the shell in the initial state, ρ is the density of the shell material and the Newton parameter $A_N=\rho_BL^3/M_S$. The dimensional value of the acceleration $\mathbf{a}=\partial^2\mathbf{r}/\partial t^2$ is expressed in terms of the dimensionless parameter $\tilde{\mathbf{a}}$ in accordance with the relation $\mathbf{a}=\tilde{\mathbf{a}}V_0^2/L$, and the substitution $t=\tau L/V_0$ is also made, where τ is the dimensionless time and

$$T_{ij}=\tilde T_{ij}T_0/L,\quad p=\tilde pT_B\big/L^2,\quad g=\tilde gV_0^2\big/L,\quad \eta=\tilde\eta T^0\big/V_0,\quad \rho=\tilde\rho M_c\big/L^2$$

The equation of motion of the element of the shell (1.12) is reduced to dimensionless form by substituting the assumed relations.

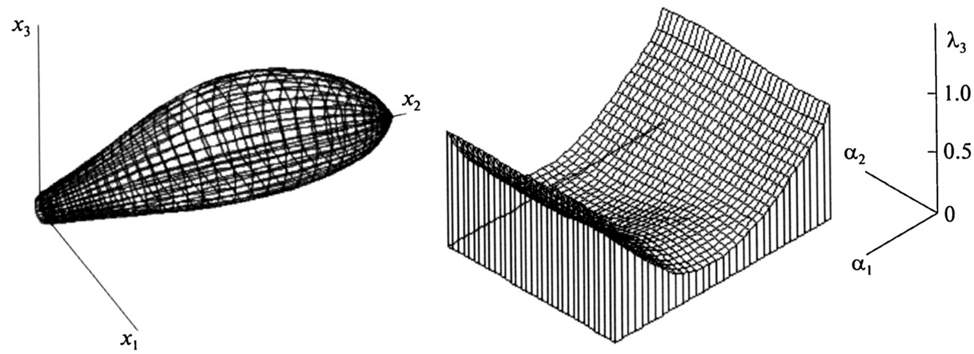

Fig. 2.

5. Examples of the calculations

The non-axisymmetric (because the motion of the shell is constrained by a plane) and the axisymmetric loading of a shell of revolution made of an elastomer (IRP-2052 rubber [3]) were considered as examples. This shell has the following initial dimensions (Fig. 1): maximum diameter $2r = 0.0292$ m, diameter of the inlet section $2r_0 = 0.0146$ m, initial length of the generatrix of the shell $L = 0.12$ m, and initial thickness of the shell material $h = 0.25 \times 10^{-4}$ m. The physical relations were taken in the form (2.3) and the coefficient of internal friction in the material $\eta = 7.5 \times 10^{-6}$ Ns/m.

An irregular mesh with 32 × 32 cells along the generatrix of the shell and the circumference respectively was used in the calculations. The calculated value of the pressure drop $p = 1570\ \mathrm{N/m^2}$, the density of air $\rho_B = 0.125\ \mathrm{kg/m^3}$, the corresponding characteristic velocity was $V_0 = 80$ m/s and the mass per unit area of the shell in the undeformed state $\rho_0 = 0.005\ \mathrm{kg/m^2}$. The problem was solved in dimensionless form as described in Section 4.

Results of the calculations. The initial shape of the shell is shown in a Cartesian system of coordinates in Fig. 1. The case of the non-axisymmetric deformation of the shell by constraining the motion of the shell with the plane $x_3 = -r$ during its filling (the left-hand part of Fig. 2) was considered in order to try out the algorithm. The introduction of this plane into the treatment corresponds to the case when, as the elements of the shell reach this plane, they "adhere" to it. Calculations show that the effect of the weight of the shell on its deformed state is negligibly small for the chosen loading conditions. The filling of the shell when there was no bounding plane $x_3 = -r$ was therefore investigated with axisymmetric loading. The degrees of reduction in the thickness for non-axisymmetric loading of the shell, the shape of which is shown on the left in Fig. 2, are shown on the right in Fig. 2: For clarity in unrolling the shell, the zones of the balloon vertex and the inlet section have been enlarged to a rectangle.

It was established that the intensity of the loading has a considerable effect on the filling of the shell. The results of calculations showing the changes in the magnitude of the pressure P and the rates of change of the pressure with respect to the volume dP/dV are presented in Fig. 3 for two different rates of increase in the pressure during the filling process (the solid and dashed lines). In both calculations the pressure increase is defined by a piecewise-linear law. The calculations show that the volume barely changes at the beginning of the filling. Here, the rate of the increase in the pressure with respect to the volume dP/dV is very large but it has a tendency to fall. In the case of a high rate of inflation (the solid curves), the plot of the derivative dP/dV has an inflection at $\tau \approx 0.3$, after which dP/dV continues to decrease. When $\tau \approx 1$, the increase in the volume is so large (although the pressure also continues to grow) that the shell is filled at a high rate and transfers to a critical filling state. In the case of a lower rate of inflation (the dashed lines), the duration of the filling process, up to when the thickness of the shell wall is reduced by 0.05, increases and the function dP/dV has two local maxima (at $\tau \approx 0.55$ and ≈ 1.25). The increase in the value of dP/dV up to the first maximum is associated with the fact that, when the pressure intensity is reduced, the main motion of the shell occurs along the x_2 axis (Fig. 1) and, at the same time, the volume of the shell barely changes. However, when $\tau > 0.55$, there is an intense increase in the volume of the shell. The second local maximum (when $\tau \approx 1.25$) is less distinct.

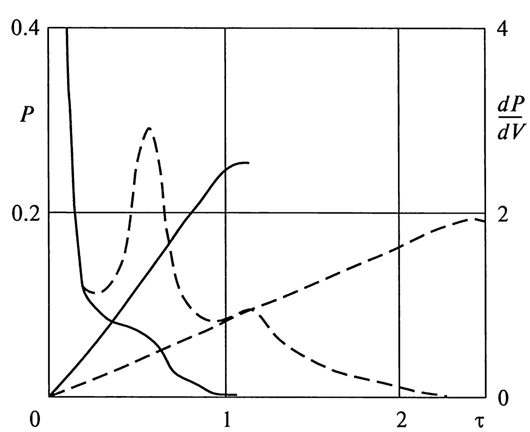

Fig. 3.

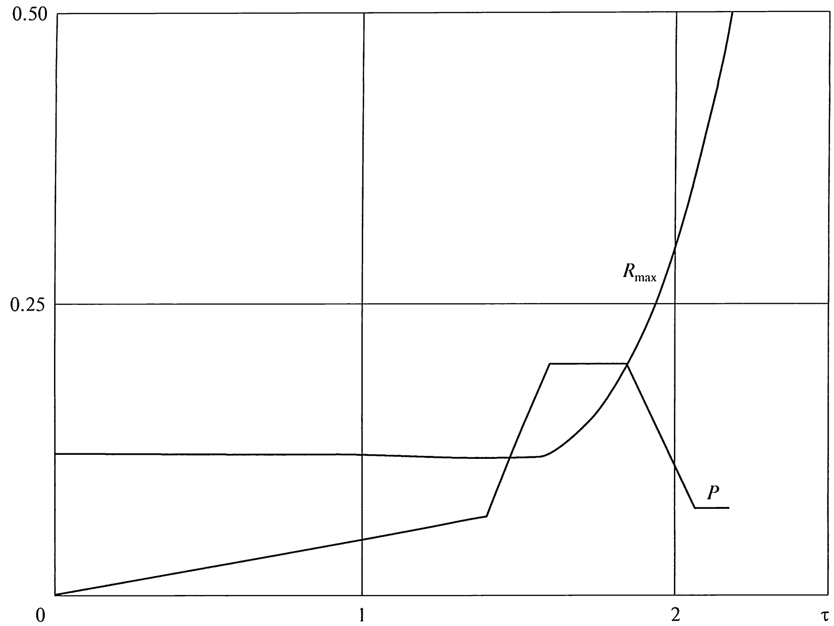

Fig. 4.

Hence, it follows from an analysis of the graphs presented in Fig. 3 that, when the rate of growth of the pressure (the angle of inclination of the graph of the relation between the pressure differential and time) decreases, the value of the critical pressure in the dynamic problem is reduced and the duration of the filling of the shell up to the specified degree of the narrowing down of its thickness of 0.05 increases.

We will now consider the critical loading state of the shell. The law for the change in the pressure differential used in calculations of the filling of the shell is shown in Fig. 4. In the pressure differential graph, the value corresponding to the horizontal shelf was chosen on the basis of numerical experiments in such a way that, below this shelf, filling occurs up to the form of an equilibrium state and, in the case of a pressure increase that is level with or above this shelf, there is unstable "critical" filling. Analysis of the graph for the calculated change in the maximum radius of the shell R_{max} with time shows that, close to the time value $\tau \approx 2$, although a reduction in the pressure differential is specified, the inflation of the shell continues intensively since the force of the pressure acting on an element of the shell is defined as the product of the pressure and the area of the deformed element and this area increases faster than the pressure differential falls which is assisted by the increasing radius of curvature of the shell.

Results of a calculation of the shape of the shell at the instants of inflation $\tau = 1.81$ (which corresponds to the shelf in the graph of the pressure in Fig. 4) and $\tau = 2.18$ (which corresponds to a degree of reduction in the wall thickness $\lambda_3 = 0.05$ at which the calculation was completed) are shown in Fig. 5. At the instant $\tau = 2.18$, the shell is close in shape to an ellipsoid of revolution, its length is 1.58 and its

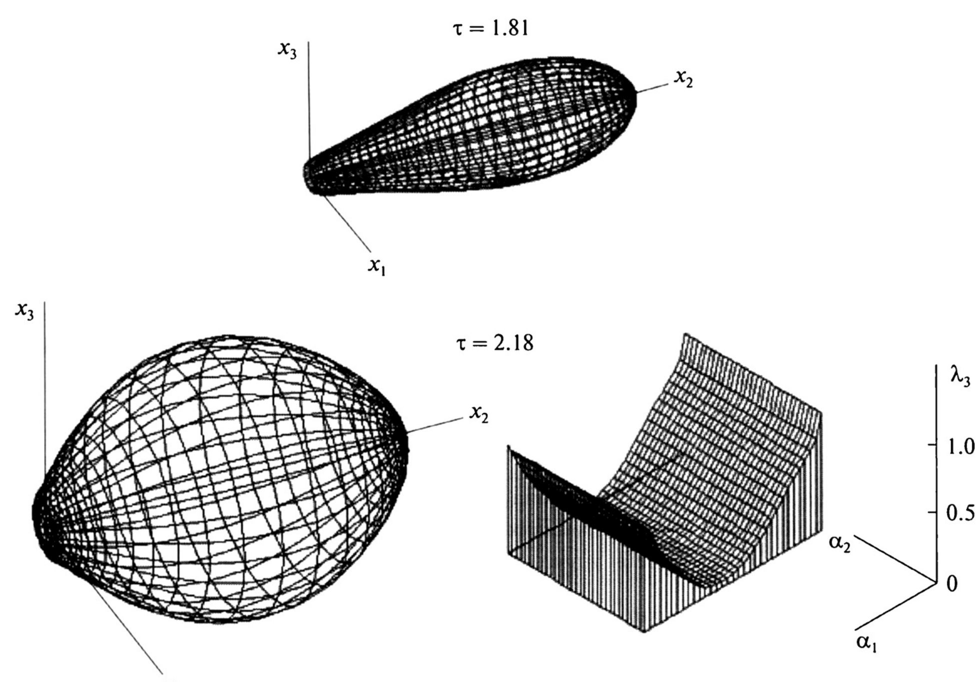

Fig. 5.

diameter 1.09 (the maximum diameter is slightly displaced towards the vertex of the shell and corresponds to the Lagrangian coordinate 0.59). The initial volume of the balloon was 0.233 units and, at the instant $\tau = 2.18$, it was 6.66 units. A diagram showing the change in the degree of narrowing of the wall of the shell along the generatrix at the instant $\tau = 2.18$ is presented in the lower right-hand part of Fig. 5. Here, as in Fig. 3, for clarity in the unrolling of the shell, the zones of the balloon vertex and the inlet section have been enlarged to a rectangle. By this time, the degree of narrowing of the wall in the middle section of the shell reaches a value $\lambda_3 = 1/(\lambda_1\lambda_2) = 0.05$.

At points on the generatrix of the shell, located at different distances starting from the inlet section, the degree of narrowing of the wall of the shell λ_3 was:

$$0.959, 0.607, 0.391, 0.168, 0.060, 0.060, 0.215, 0.496, 0.864.$$

6. Conclusion

The results of the computational experiments show that the critical value of the internal pressure at which instability of the filling process begins (that is, intense inflation of the shell as the internal pressure decreases), depends on the rate of filling of the shell: the greater this rate, the greater the critical value of the pressure turns out to be, which is also observed when blowing up rubber balloons.

References

1. Novozhilov VV. *Fundamentals of the Non-Linear Theory of Elasticity*. Leningrad: Gostekhizdat; 1948.
2. Galimov KZ. *Foundations of the Non-Linear Theory of Thin Shells*. Kazan: Kazan Univ; 1975.
3. Chernykh KF. *Non-Linear Theory of Elasticity in Machine Construction Calculations*. Leningrad: Mashinostroyeniye; 1986.
4. Giniyatullin AG, Gimadiyev RSh. Investigation of the filling of a balloon catheter shell. *Meditsin Tekhnika* 1993;**2**:30–3.
5. Ridel VV, Gulin BV. *The Dynamics of Soft Shells*. Moscow: Nauka; 1996.
6. Shagidullin RR. *Problems in the Mathematical Modelling of Soft Shells*. Kazan: Izd Kazan Mat Obshchestva; 2001.
7. Gimadiyev RSh. *The Dynamics of Soft Shells of the Parachute Type*. Kazan: Izd Kazan Gos Energ Univ; 2006.
8. Paimushin VN. The equations of the geometrically non-linear theory of elasticity and momentless shells for arbitrary displacements. *J Appl Math Mech* 2008;**72**(5):597–610.
9. Paimushin VN. Investigation of the equations in the theory of elasticity and plasticity for arbitrary displacements and deformations. *Izv Ross Akad Nauk MTT* 2011;**2**:67–80.
10. Berezhnoi DV, Paimushin VN. Two formulations of elastoplastic problems and the theoretical determination of the location of neck formation in samples under tension. *J Appl Math Mech* 2011;**75**(4):447–62.
11. Paimushin VN. A theory of thin shells with finite displacements and deformations based on a modified Kirchhoff–Love model. *J Appl Math Mech* 2011;**75**(5):568–79.
12. Gimadiev R. Sh., Gimadieva T.Z., Paimushin VN. Mathematical simulation of dynamics at elastomeric shells inflation//Int. Conf. Methods of Aerophysical Research. Kazan, Russia. 2012. Pt I. P. 114-115.
13. Otto F, Trostel R. *Zugbeanspruchte Konstruktionen. Bd. 1. Pneumatische Konstruktionen*. Frankfurt, Berlin: Ullstein Fachverlag; 1962.
14. Magula VE. *Elastic Marine Constructions*. Leningrad: Sudostroyeniye; 1978.
15. Karnaukhov VG, Kirichok IF. *Coupled Problems in the Theory of Viscoelastic Plates and Shells*. Kiev: Naukova Dumka; 1986.
16. Samarskii AA. *Popov YuP. Difference Methods for Solving Problems in Gas Dynamics*. Moscow: Nauka; 1992.

Translated by E. L. S.